A few exercises in analysis, probability, and combinatorics

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Contents

CHAPTER 1

ANALYSIS

1.1 Measure theory

Exercise 1.1.1. Give an example of a set E, a σ-algebra $\mathscr{A}$ on E and an application $f\colon E \to F$ such that

$$\left\{f(A)\colon A \in \mathscr{A}\right\}$$

is *not* a σ-algebra on $f(E)$.

Exercise 1.1.2. Let

$$\mathscr{C} := \left\{[a,b)\colon a,b \in \mathbb{Q},\ a < b\right\}.$$

Prove that the σ-algebra $\sigma(\mathscr{C})$ generated by $\mathscr{C}$ is the Borel σ-algebra $\mathscr{B}(\mathbb{R})$ of $\mathbb{R}$.
Hint. Recall that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Exercise 1.1.3. Let E, F be two sets, $\mathscr{A}$ and $\mathscr{B}$ two σ-algebras on E and F respectively and $f\colon E \to F$ an application. Recall the notion of *inverse image*

$$f^{-1}\langle B\rangle := \left\{x \in E\colon f(x) \in B\right\}$$

of a subset $B \subseteq F$ by f.

1. Prove that the set of subsets of E defined by

   $$f^{-1}\langle\mathscr{B}\rangle := \left\{f^{-1}\langle B\rangle\colon B \in \mathscr{B}\right\}$$

   is a σ-algebra on E.

2. Prove that the set of subsets of F defined by

   $$f[\mathscr{A}] := \left\{B \subseteq F\colon f^{-1}\langle B\rangle \in \mathscr{A}\right\}$$

   is a σ-algebra on F.

Exercise 1.1.4. Let $E := [0,1)$, $n \in \{1,2,\ldots\}$ and $0 =: a_0 < a_1 < \cdots < a_n := 1$. Give $\sigma(\mathscr{C})$, the smallest σ-algebra on E which contains all elements of

$$\mathscr{C} := \Big\{[a_{i-1}, a_i) \colon 1 \leqslant i \leqslant n\Big\}.$$

Exercise 1.1.5. Let E be a set. Show that

$$\mathscr{A} := \Big\{A \subseteq E \colon A \text{ or } E \setminus A \text{ is finite or countably infinite}\Big\}$$

is a σ-algebra on E.

Exercise 1.1.6. Let E be a set, $\mathscr{A}$ a σ-algebra on E and μ, ν two measures on $(E, \mathscr{A})$ such that $\mu(E) = \nu(E) = 1$. Prove that the set

$$\mathscr{D} := \Big\{A \in \mathscr{A} \colon \mu(A) = \nu(A)\Big\}$$

is a Dynkin system.

Exercise 1.1.7. Let E be a set and $\mathscr{A}$ a σ-algebra on E. We suppose that $\mathscr{A}$ is finite or countably infinite. For x in E we define

$$A(x) := \bigcap_{\substack{A \in \mathscr{A} \\ \text{s.t. } x \in A}} A.$$

1. Show that $A(x) \in \mathscr{A}$ and that $A(x)$ is the smallest element of $\mathscr{A}$ which contains x. (Prove that for all $A \in \mathscr{A}, \quad x \in A \implies A(x) \subseteq A$.)
2. Show that for all $x, y \in E, \quad y \in A(x) \implies A(x) = A(y)$.

   Hint. Use that $E \setminus A(y) \in \mathscr{A}$.
3. Let $\mathscr{E} := \{B \subseteq E \colon \exists x \in E,\ B = A(x)\}$. Prove that $\mathscr{A} = \sigma(\mathscr{E})$.
4. Let $\mathscr{P}(\mathscr{E})$ denote the powerset of $\mathscr{E}$. Show that the application

$$\begin{aligned} \Phi \colon \mathscr{P}(\mathscr{E}) &\longrightarrow \mathscr{A} \\ \mathscr{B} &\longmapsto \bigcup_{B \in \mathscr{B}} B \end{aligned}$$

   is injective.

Remark. This exercise proves that there is no countably infinite σ-algebra (as the powerset of any set cannot be countably infinite).

Exercise 1.1.8. Let $(\Omega, \mathscr{A}, \mu)$ be a measured space. We write

$$\mathscr{N}_\mu := \Big\{N \subseteq \Omega \colon \exists B \in \mathscr{A},\ N \subseteq B \text{ and } \mu(B) = 0\Big\}$$

for the set of μ-negligible subsets of Ω. Recall also the completion of $\mathscr{A}$ w.r.t. μ:

$$\mathscr{A}_\mu := \Big\{A \subseteq \Omega \colon \exists (E,F) \in \mathscr{A}^2,\ E \subseteq A \subseteq F \text{ and } \mu(F \setminus E) = 0\Big\}.$$

It is known that $\mathscr{A}_\mu \supseteq \mathscr{A}$ is still a σ-algebra on Ω.
Show that $\mathscr{A}_\mu = \{A \subseteq \Omega \colon \exists (E,N) \in \mathscr{A} \times \mathscr{N}_\mu,\ A = E \cup N\}$.

Exercise 1.1.9. Let $(\Omega, \mathscr{A}, \mu)$ be a measure space and $f_n\colon \Omega \to [-\infty, \infty]$, $n \in \mathbb{N}$, be a sequence of measurable functions such that

$$f(\omega) := \lim_{n\to\infty} f_n(\omega)$$

exists for μ-almost every $\omega \in \Omega$. We denote by D the domain of the function f.

1. Recall briefly why $D \in \mathscr{A}$ and $f\colon D \to [-\infty, \infty]$ is measurable.
2. Recall what "μ-almost every" means in general, and here in terms of $\Omega \setminus D$.
3. We suppose that $f_n \geqslant 0$ for all $n \in \mathbb{N}$, and that the limit

$$L := \lim_{n\to\infty} \int f_n \,\mathrm{d}\mu$$

exists in $[0, \infty)$.

   a) What can you say about $\displaystyle\int f \,\mathrm{d}\mu$? Does it exist, is it finite? What if $L = 0$?
   b) Show with the help of a counterexample that in general, $\displaystyle\int f_n \,\mathrm{d}\mu \not\longrightarrow \int f \,\mathrm{d}\mu$.
   c) What additional *sufficient* condition on (f_n) would imply $\displaystyle\int f_n \,\mathrm{d}\mu \longrightarrow \int f \,\mathrm{d}\mu$?

4. We no longer make the assumptions of Question 3, and suppose instead that f_n is integrable for every $n \in \mathbb{N}$.
   a) Show with the help of a counterexample that f is not necessarily integrable.
   b) What additional *sufficient* condition on the sequence (f_n) would guarantee both the integrability of f and the convergence $\displaystyle\int f_n \,\mathrm{d}\mu \longrightarrow \int f \,\mathrm{d}\mu$?

Exercise 1.1.10. Let λ_2 denote the Lebesgue measure on $(\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2))$, and

$$D := \{(s,s)\colon s \in (0,1)\}, \quad E := \{(s, s+t)\colon s, t \in (0,1)\}.$$

Justify that $D, E \in \mathscr{B}(\mathbb{R}^2)$ and use the translation invariance of λ_2 to show that

1. $\lambda_2(D) = 0$,
2. $\lambda_2(E) = 1$.

Exercise 1.1.11 (True or false?). Let λ denote the Lebesgue measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. Prove or disprove (with a counterexample) the following statements:

1. Let $A \in \mathscr{B}(\mathbb{R})$.
   a) If $B \subseteq A$ then $B \in \mathscr{B}(\mathbb{R})$.
   b) If $\lambda(A) = \infty$ then A is an unbounded set.

c) If $\lambda(A) < \infty$ then A is a bounded set.

d) If $\lambda(A) = 0$ then A is a bounded set.

e) If A is an open set then $\lambda(A) > 0$.

f) If $\lambda(A \cap (0,1)) = 1$ then $A \cap (0,1)$ is dense in $(0,1)$.

g) If $A \cap (0,1)$ is dense in $(0,1)$ then $\lambda(A \cap (0,1)) > 0$.

h) If $\lambda(A) > 0$ then A has a non-empty interior.

2. (In the following statements, measurability is meant w.r.t. the Borel σ-field.)

a) If $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable, then f' is measurable.

b) If $f_1, f_2, \ldots \colon \mathbb{R} \to \mathbb{R}$ are measurable functions, then the set $B := \{x \in \mathbb{R} \colon \lim_{n\to\infty} f_n(x) \text{ exists}\}$ is measurable.

c) If $f\colon [0,1] \to \mathbb{R}$ is such that $\{x \in [0,1] \colon f(x) = c\}$ is measurable for all $c \in \mathbb{R}$, then f is measurable.

Exercise 1.1.12. Let λ_n denote the Lebesgue measure on $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$. Show that for any hyperplane $H \subset \mathbb{R}^n$, $\lambda_n(H) = 0$.
Hint. Show first $\lambda_n(H_0) = 0$ for the hyperplane $H_0 := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \colon x_n = 0\}$.

Exercise 1.1.13. Let $(\Omega, \mathscr{A}, \mu)$ be a measure space and $f\colon \Omega \to [-\infty, \infty]$.

1. We suppose that $f \in \mathrm{L}^1(\Omega, \mathscr{A}, \mu)$. Show that $|f(\omega)| < \infty$ for μ-a.e. $\omega \in \Omega$.

2. We suppose that there is a sequence f_n, $n \in \mathbb{N}$, converging to f in $\mathrm{L}^1(\Omega, \mathscr{A}, \mu)$. Show that there is a subsequence (f_{n_k}) of (f_n) converging to f μ-a.e., that is
$$\lim_{k\to\infty} f_{n_k}(\omega) = f(\omega)$$
for μ-a.e. $\omega \in \Omega$.

Exercise 1.1.14. Let $a \in \mathbb{C}$ with $|a| < 1$. Show that the two sums
$$\sum_{n=1}^{\infty} \frac{a^n}{1 - a^{2n}} \qquad \text{and} \qquad \sum_{m=1}^{\infty} \frac{a^{2m-1}}{1 - a^{2m-1}}$$
are well defined and equal.
Hint. Introduce $f_{n,m} := a^{n(2m-1)}$ for $m, n \in \mathbb{N}$ and apply Fubini's theorem.

Exercise 1.1.15. Let $(\Omega, \mathscr{A}, \mu)$ be a measure space.

1. Let $n \in \mathbb{N}$ and $f_1, \ldots, f_n \in \mathrm{L}^n(\Omega, \mathscr{A}, \mu)$. Show that
$$\|f_1 \cdots f_n\|_1 \leqslant \prod_{i=1}^{n} \|f_i\|_n.$$
Hint. Proceed by induction and recall Hölder's inequality.

2. We suppose here that $\mu = \mathbb{P}$ is a probability measure (*i.e*, $\mathbb{P}(\Omega) = 1$). Show that for every finite family $\{A_1, \dots, A_n\} \subseteq \mathscr{A}$ of events on Ω,

$$\mathbb{P}(A_1 \cap \cdots \cap A_n) \leqslant \left[\mathbb{P}(A_1) \cdots \mathbb{P}(A_n)\right]^{1/n}.$$

Remark. By comparison between arithmetic and geometric means, this inequality is sharper than the (trivial) inequality

$$\mathbb{P}(A_1 \cap \cdots \cap A_n) \leqslant \frac{\mathbb{P}(A_1) + \cdots + \mathbb{P}(A_n)}{n}.$$

Exercise 1.1.16. Let $(\Omega, \mathscr{A}, \mu)$ be a measure space. We suppose that there exists a measurable function $f : \Omega \to (0, \infty)$ such that f and $1/f$ are integrable (w.r.t. μ). Prove that μ is finite.

Exercise 1.1.17. Let $(\Omega, \mathscr{A}, \mu)$ be a σ-finite measure space and $f : \Omega \to [0, \infty]$ be a measurable function. Let $E_t := \{\omega \in \Omega : f(\omega) > t\}$ for each $t > 0$. Prove that

$$\int_\Omega f \, \mathrm{d}\mu = \int_{(0,\infty)} \mu(E_t) \, \mathrm{d}\lambda_1(t).$$

Exercise 1.1.18.

1. Compute the double integral
$$\iint\limits_{(0,\infty)^2} \frac{\mathrm{d}\lambda_2(x,y)}{(1+y)(1+yx^2)}.$$

2. Deduce that
$$\int_{(0,\infty)} \frac{\log x}{x^2 - 1} \, \mathrm{d}\lambda_1(x) = \frac{\pi^2}{4}.$$

   Hint. Observe that $\dfrac{1}{(1+y)(1+x^2 y)} = \dfrac{1}{x^2-1}\left(\dfrac{x^2}{1+yx^2} - \dfrac{1}{1+y}\right)$.

3. Show that
$$\int_{(0,1)} \frac{\log x}{x^2 - 1} \, \mathrm{d}\lambda_1(x) = \frac{\pi^2}{8}.$$

Exercise 1.1.19. Let $f : \mathbb{R}^2 \to [0, \infty)$ be a measurable function, and

$$I := \iint\limits_{(0,1)^2} f\left(\sqrt{-2\log u}\cos(2\pi v), \sqrt{-2\log u}\sin(2\pi v)\right) \mathrm{d}\lambda_2(u,v).$$

Show that

$$I = \iint\limits_{\mathbb{R}^2} f(x,y) \, \frac{e^{-\frac{x^2+y^2}{2}}}{(\sqrt{2\pi})^2} \, \mathrm{d}\lambda_2(x,y).$$

Exercise 1.1.20.

1. Let $t > 0$. Show that
$$\int_{(0,t)} \frac{\sin x}{x} \,\mathrm{d}\lambda_1(x) = \int_{(0,\infty)} \left(\int_{(0,t)} e^{-xy} \sin x \,\mathrm{d}\lambda_1(x) \right) \mathrm{d}\lambda_1(y).$$

2. Deduce that
$$\int_{(0,t)} \frac{\sin x}{x} \,\mathrm{d}\lambda_1(x) = \int_{(0,\infty)} \frac{1 - e^{-ty}(y \sin t + \cos t)}{1 + y^2} \,\mathrm{d}\lambda_1(y)$$
for all $t > 0$, and conclude that
$$\lim_{t\to\infty} \int_{(0,t)} \frac{\sin x}{x} \,\mathrm{d}\lambda_1(x) = \frac{\pi}{2}.$$
Hint. Apply (*properly!*) the dominated convergence theorem.

3. Is the function $x \mapsto \dfrac{\sin x}{x}$ Lebesgue-integrable on $(0,\infty)$?

Exercise 1.1.21. Let $(\Omega, \mathscr{A}, \mu)$ be a measure space and $f : \Omega \to \mathbb{R}$ be a measurable function. For $p \in [1,\infty]$, we set $\|f\|_p := \infty$ if $f \notin \mathrm{L}^p(\Omega, \mathscr{A}, \mu)$.

1. Let $1 \leqslant p < q \leqslant \infty$ and suppose *for this question only* that $\mu(\Omega) < \infty$. Show that
$$\|f\|_p \leqslant \mu(\Omega)^{\frac{1}{p} - \frac{1}{q}} \|f\|_q$$
(with the convention $1/\infty = 0$).
Remark. We get $\mathrm{L}^q(\Omega, \mathscr{A}, \mu) \subset \mathrm{L}^p(\Omega, \mathscr{A}, \mu)$ (under the above conditions).

2. Suppose that $f \in \mathrm{L}^r(\Omega, \mathscr{A}, \mu)$ for some $1 \leqslant r < \infty$. Prove that
$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty.$$
Hint. Show $\liminf_{p\to\infty} \|f\|_p \geqslant \|f\|_\infty$ using Chebyshev's inequality.

Exercise 1.1.22. Suppose that $(\Omega, \mathscr{A}) := (\mathbb{Z}, \mathscr{P}(\mathbb{Z}))$ and μ is the counting measure on $\mathbb{Z}$ and consider the sequence space $\ell^p := \mathrm{L}^p(\Omega, \mathscr{A}, \mu)$ for $p \in [1,\infty]$. As above, we set $\|f\|_p := \infty$ if $f \notin \ell^p$. Show that
$$\|f\|_q \leqslant \|f\|_p$$
whenever $1 \leqslant p < q \leqslant \infty$. In particular there is the inclusion $\ell^p \subset \ell^q$.

Exercise 1.1.23. Let $p \neq q$ in $[1,\infty]$. Prove that $\mathrm{L}^p(\mathbb{R}) \setminus \mathrm{L}^q(\mathbb{R}) \neq \emptyset$.

Exercise 1.1.24 (Riesz–Scheffé's lemma). Let $(\Omega, \mathscr{A}, \mu)$ be a measure space, and $f, f_1, f_2, \ldots \in \mathrm{L}^p(\Omega)$ with $p \in [1,\infty)$. We suppose that, as $n \to \infty$, $f_n(\omega) \to f(\omega)$ for μ-a.e. $\omega \in \Omega$ and that $\|f_n\|_p \to \|f\|_p$. Let $\mathrm{sign} : \mathbb{R} \to \{-1, 1\}$ denote a function such that $|x| = (\mathrm{sign}\, x)x$ for all $x \in \mathbb{R}$, and write
$$f_n^* := f_n \mathbb{1}_{\{|f_n| \leqslant |f|\}} + (\mathrm{sign}\, f_n)|f| \mathbb{1}_{\{|f_n| > |f|\}}$$
for every $n \in \mathbb{N}$.

1. Show that $\|f_n^* - f\|_p \to 0$ as $n \to \infty$.
2. Show that $\|f_n - f_n^*\|_p \to 0$ as $n \to \infty$. Conclude that $f_n \to f$ in $\mathrm{L}^p(\Omega, \mathscr{A}, \mu)$.

   Hint. Use the convexity inequality $(y-x)^p \leqslant y^p - x^p$ for $0 \leqslant x \leqslant y$.

Exercise 1.1.25. If $f: \mathbb{R} \to \mathbb{R}$ is measurable and $h \in \mathbb{R}$, we define $\tau_h f: x \mapsto f(x+h)$ "the translation of f by h" which is obviously also measurable. Let $1 \leqslant p < q \leqslant \infty$ such that $1/p + 1/q = 1$, $f \in \mathrm{L}^p(\mathbb{R})$ and $g \in \mathrm{L}^q(\mathbb{R})$. Recall that the convolution $f \star g$ of f and g is given by

$$f \star g(x) := \int_{\mathbb{R}} f(y) g(x-y)\, \lambda_1(\mathrm{d}y), \qquad \lambda_1\text{-a.e.} \tag{$\star$}$$

1. Show that $\tau_h f \to f$ in $\mathrm{L}^p(\mathbb{R})$ as $h \to 0$.

   Hint. Approximate f smoothly; note that $p < \infty$.
2. In the special case $p = 1$ (so $q = \infty$), show that the definition in ($\star$) is actually valid *everywhere* and makes $f \star g$ be a bounded and uniformly continuous function.

Exercise 1.1.26. Let λ denote the Lebesgue measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. Recall that, by translation invariance of λ, for any $E \in \mathscr{B}(\mathbb{R})$ with $\lambda(E) > 0$ the set

$$E - E := \{x - y:\ x, y \in E\}$$

contains some open interval centered at 0: $\exists \varepsilon > 0,\ (-\varepsilon, \varepsilon) \subset E - E$. In this exercise we suppose that $f: \mathbb{R} \to \mathbb{R}$ is a measurable function such that

$$\forall (x, y) \in \mathbb{R}^2, \quad f(x+y) = f(x) + f(y). \tag{$\star$}$$

1. For $k \in \mathbb{N}$, justify that the set $E_k := \{x \in \mathbb{R}:\ |f(x)| < k\}$ is in $\mathscr{B}(\mathbb{R})$ and, by observing the identity

   $$\bigcup_{k \in \mathbb{N}}\uparrow E_k = \mathbb{R},$$

   show that there exist $k \in \mathbb{N}$ and $\varepsilon > 0$ such that: $|x| < \varepsilon \implies |f(x)| < 2k$.
2. Deduce that $f(x) \to 0$ as $x \to 0$.
3. Conclude that $f(x) = f(1)x$ for all $x \in \mathbb{R}$.

   Hint. Use the density of $\mathbb{Q}$ in $\mathbb{R}$.

1.2 Linear differential equations

Exercise 1.2.1. Solve (over $\mathbb{R}$) the following systems of linear differential equations:

1. $\begin{cases} x' = x + z \\ y' = -y - z \\ z' = 2y + z \end{cases}$

2. $\begin{cases} x' = 2x - y + 4t \\ y' = x + e^{-t} \end{cases}$

3. $\begin{cases} x' = \cos(t)x - \sin(t)y \\ y' = \sin(t)x + \cos(t)y \end{cases}$

   Hint. Rewrite the system as a first order differential equation in $z := x + iy$.

Exercise 1.2.2. We consider the following Cauchy problem.

$$(1+t^2)x'' - t(1-t^2)x' + (1-t^2)x = 0, \qquad \text{(E)}$$
$$x(0) = 1, \quad x'(0) = 1.$$

1. Show that the functions $t \mapsto At$, $A \in \mathbb{R}$, are solutions to (E) but that none of them is a solution to the Cauchy problem.

2. Find all solutions to (E) by letting the constant A vary with t.

   Hint:
   $$-\frac{2+t^2+t^4}{t(1+t^2)} = \frac{2t}{1+t^2} - t - \frac{2}{t}.$$

   Now, solve the Cauchy problem.

Exercise 1.2.3. The goal is to find all twice differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 1$ and

$$\forall (s,t) \in \mathbb{R}^2, \quad f(s+t) + f(s-t) = 2f(s)f(t).$$

Let f be such a function.

1. Show that f is an even function.

2. Show that f is a solution to $x'' = \lambda x$ for some constant $\lambda \in \mathbb{R}$.

3. Conclude.

Exercise 1.2.4. Let $A(t) := (a_{i,j}(t)) \in \mathbb{R}^{n\times n}$ be a matrix, and $X_1(t), \dots, X_n(t) \in \mathbb{R}^n$ be n solutions to the linear differential equation

$$X'(t) = A(t)X(t). \qquad \text{(F)}$$

We define

$$W(t) := \Big[X_1(t) \,\Big|\, X_2(t) \,\Big|\, \cdots \,\Big|\, X_n(t) \Big] \in \mathbb{R}^{n\times n}$$

and

$$w(t) := \det(W(t)) = \det(W_1(t), \dots, W_n(t)),$$

where $W_1(t), \dots, W_n(t)$ are the rows of the matrix $W(t)$.

1. Recalling that the determinant is a multilinear form, prove that

$$w'(t) = \sum_{i=1}^{n} \det\Big(W_1(t), \dots, W_{i-1}(t), W_i'(t), W_{i+1}(t), \dots, W_n(t)\Big).$$

   Check also that $W_i'(t) = \sum_{j=1}^{n} a_{i,j}(t) W_j(t)$ for every $i = 1, \dots, n$.

2. Recalling that the determinant is an alternating form, deduce that w is a solution to the homogeneous, first order, linear differential equation

$$y' = \operatorname{tr}(A(t))\, y.$$

3. Prove that either $(\forall\, t \in \mathbb{R},\ w(t) = 0)$ or $(\forall\, t \in \mathbb{R},\ w(t) \neq 0)$, and that the latter happens if and only if $(X_1, X_2, \dots, X_n)$ is a basis of solutions to (F).

   Hint. Recall the isomorphism $X \mapsto X(0)$ from the solutions to (F) onto $\mathbb{R}^n$.

CHAPTER 2

PROBABILITY

2.1 Combinatorial probability

Exercise 2.1.1. In an urn, there are 17 green, 5 blue, and 11 red, indistinguishable balls. Answer the following questions (specify in each case the probability space):

1. We pick two balls simultaneously (without replacement). What is the probability that none of these balls is red?

2. We pick three balls one after the other, with replacement. What is the probability that at most two of these balls are green?

Exercise 2.1.2. We consider a 5-card hand from a traditional deck of 52 cards. Specify the probability space and find the probability that the hand contains...

1. five cards of the same suit;

2. four cards of the same rank;

3. five cards of sequential rank (the aces having both the lowest and highest ranks);

4. three cards of the same rank and two other cards of another rank.

Exercise 2.1.3. Let X be a Poisson random variable with parameter $\lambda > 0$, that is

$$\mathbb{P}(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}, \qquad \text{for } k = 0, 1, 2, \ldots$$

1. Show that $\mathbb{E}[X] = \lambda$.

2. Show that $\mathrm{Var}(X) = \lambda$.

Exercise 2.1.4. Let $n \in \mathbb{N}$, $x \in [0,1]$ and X_n be a random variable having the binomial distribution with parameter (n,p), that is

$$\mathbb{P}(X_n = k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad k = 0,1,2,\dots,n.$$

1. Show that $\mathbb{E}[X_n] = np$.
2. Show that $\mathrm{Var}(X_n) = np(1-p)$.

Exercise 2.1.5. Show that $\mathscr{C} := \{[a,b) : a,b \in \mathbb{Q}\}$ generates the Borel σ-algebra $\mathscr{B}(\mathbb{R})$ of $\mathbb{R}$.

Exercise 2.1.6. Let $\mathscr{C} := \{C_i\}_{1 \leqslant i \leqslant n}$ be a finite partition of Ω, *i.e*, $\Omega = \bigcup_{i=1}^n C_i$ with $C_1, \dots, C_n$ all non-empty and pairwise disjoint. Describe $\sigma(\mathscr{C})$, the smallest σ-algebra containing $\mathscr{C}$.

Exercise 2.1.7. Let $(\Omega, \mathscr{A}, P)$ be a probability space.

1. Let A, B be two events, and its *symmetric difference* $A \Delta B := (A \cup B) \setminus (A \cap B)$. Prove using the axioms of probability that
$$|P(A) - P(B)| \leqslant P(A \Delta B).$$
2. Let $A_n, n \geqslant 1$, be a sequence of events with $P(A_n) = 1$ for every n. Prove that
$$P\left(\bigcap_{n \geqslant 1} A_n\right) = 1.$$

Exercise 2.1.8. Let X be a random variable with values in $\mathbb{N}$. Prove that

$$\mathbb{E}[X] = \sum_{\ell=1}^{\infty} \mathbb{P}(X \geqslant \ell)$$

(with the convention that $\mathbb{E}[X] = \infty$ in case the first moment of X does not exist).

2.2 Distributions, independence

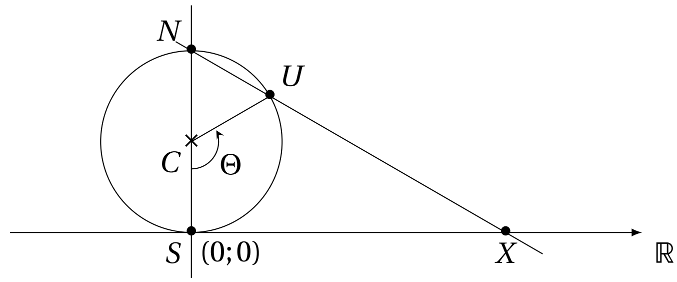
Exercise 2.2.1. Suppose a distribution function F is given by

$$F(x) = \frac{1}{4}\mathbb{1}_{[0,\infty)}(x) + \frac{1}{2}\mathbb{1}_{[1,\infty)}(x) + \frac{1}{4}\mathbb{1}_{[2,\infty)}(x).$$

Let P be the probability measure, $P((-\infty, x]) := F(x)$, $x \in \mathbb{R}$. Find the probability of:

$$A = (-12, 12), \quad B = (-12, 32), \quad C = (23, 52), \quad D = [0, 2), \quad E = (3, \infty).$$

Exercise 2.2.2. For each point $U \neq N$ on the circle with center $C(0;1/2)$ and diameter 1 below, the line (NU) intersects the real axis at a unique point — we call X its abscissa:



We suppose that U has a uniform distribution, namely we consider that the measure Θ of the oriented angle $(\overrightarrow{CS};\overrightarrow{CU})$ is uniformly distributed on $(-\pi,\pi)$.
Show that X has the standard Cauchy distribution.

Exercise 2.2.3. Let X, Y be two independent Bernoulli(1/2) r.v. and $Z := \frac{1}{2}\big(1+(-1)^{X+Y}\big)$.

1. Show that Z is a Bernoulli(1/2) r.v. which is independent of X and of Y.
2. Check that Z is *not* independent of (X, Y).

Exercise 2.2.4.

1. Let $X_1, X_2, \dots$ be identically distributed real r.v. and N be a $\mathbb{N}_0$-valued r.v. We suppose N and X_ℓ independent for each $\ell \in \mathbb{N}$, and that $\mathbb{E}[|X_1|] < \infty$, $\mathbb{E}[N] < \infty$. Let the random sum
$$S(\omega) := \sum_{\ell=1}^{N(\omega)} X_\ell(\omega), \qquad \omega \in \Omega.$$
Show that S is integrable and $\mathbb{E}[S] = \mathbb{E}[N]\,\mathbb{E}[X_1]$.
Hint. Recall Exercise 2.1.8.
2. With one initial bet of 50 CHF, you are allowed to roll two fair traditional dice. Each time the sum of the two faces up is greater than or equal to 7, you win either 30 CHF or 40 CHF depending on the result of a fair coin toss, and moreover you can roll the dice again. If however the sum is less than 7, then the game is over. Is this game favorable to you?

Exercise 2.2.5. For any real r.v. X, let F_X denote its cumulative distribution function.

1. Check that $\lim_{t\to-\infty} F_X(t) = 0$ and $\lim_{t\to\infty} F_X(t) = 1$.
2. Let X and Y be two *independent* r.v. having the exponential distribution with rates $\lambda > 0$ and $\mu > 0$ respectively, *e.g.*
$$F_X(t) = \begin{cases} 1 - e^{-\lambda t}, & \text{if } t \geqslant 0, \\ 0, & \text{otherwise.} \end{cases}$$
   a) Let $\theta > 0$. Show that θX has the exponential distribution with rate λ/θ.
   b) Show that $Z := \min(X, Y)$ has the exponential distribution with rate $\lambda + \mu$.

3. Let $X, X_1, X_2, \dots$ be i.i.d. real r.v. We suppose that for every $n \in \mathbb{N}$, the r.v. $Z_n := n \min(X_1, \dots, X_n)$ has the same law as X and we note $S := 1 - F_X$.

   a) Show that $S(nt) = S(t)^n$ for every $n \in \mathbb{N}$ and every $t \in \mathbb{R}$.

   b) Deduce that $\mathbb{P}(X < 0) = 0$, and $S(r) = S(1)^r$ for every rational $r > 0$.

   c) Show that if $S(1) = 0$, then $\mathbb{P}(X = 0) = 1$.

   d) Assume now $S(1) \neq 0$. Show then that $0 < S(1) < 1$, and conclude that X has the exponential distribution with rate $\log(1/S(1))$.

Exercise 2.2.6. Let A and B be two points picked independently and uniformly inside the unit disk $D := D(0;1)$. Write $Z := |AB|$ for the distance between A and B. Find the probability that the disk $D(A, Z)$ with center A and radius Z lies inside D.

Exercise 2.2.7. Let X be a geometric random variable with parameter $p \in [0,1]$, that is

$$\mathbb{P}(X = k) = (1-p)^{k-1} p, \qquad k = 1, 2, \dots$$

1. Compute the c.d.f. of X.

2. Let $q \in [0,1]$ and Y be a Geometric(q) random variable independent of X. Show that $Z := \min(X, Y)$ has the geometric distribution with parameter $1 - (1-p)(1-q)$.

Exercise 2.2.8.

1. Give an example of c.d.f. having an infinite number of discontinuities.

2. Show that every c.d.f. has at most countably many discontinuities.

3. Let X, Y be random variables with c.d.f. F, G respectively, and B be a Bernoulli(1/2) r.v. independent of X and of Y. Compute the c.d.f. of $Z := BX + (1-B)Y$.

Exercise 2.2.9 (True or false?). Prove, or disprove (by giving a counterexample), briefly the following statements. We consider real r.v. on some general probability space $(\Omega, \mathscr{A}, \mathbb{P})$.

1. *About the laws of random variables.*

   a) For every measurable function $f : \mathbb{R} \to \mathbb{R}$,

   b) If $\mathbb{P}(X = t) = \mathbb{P}(Y = t)$ for all $t \in \mathbb{R}$, then $\mathbb{P}(X = Y) = 1$.

   c) If $\mathbb{P}(X \leqslant t) = \mathbb{P}(Y \leqslant t)$ for all $t \in \mathbb{R}$, then $\mathbb{P}(X = Y) = 1$.

   d) If $\mathbb{P}(X = t) = \mathbb{P}(Y = t)$ for all $t \in \mathbb{R}$, then X and Y have the same law.

   e) If X and Y have same law and $X \geqslant 0$ a.s., then $Y \geqslant 0$ a.s.

   f) If X and Y have same law, then $\mathbb{P}(X < Y) = \mathbb{P}(X > Y)$.

   g) If X and Y have same law and $X \in \mathrm{L}^1(\mathbb{P})$, then $Y \in \mathrm{L}^1(\mathbb{P})$ and $\mathbb{E}[X] = \mathbb{E}[Y]$.

   h) If X and Y have same law, then $X + Z$ and $Y + Z$ also have same law.

2. *About independence.*

   a) If $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$, then X and Y are independent.

   b) If X and Y are independent, then $\mathbb{P}(X = Y) = 0$.

   c) If X and Y are independent, then $\mathbb{P}(X = Y) < 1$.

   d) If Z is independent of both X and Y, then Z is independent of (X, Y).

   e) If X, Y are independent, then so are $f(X), g(Y)$ for $f, g\colon \mathbb{R} \to \mathbb{R}$ measurable.

2.3 Computing distributions

Exercise 2.3.1. Let $p \in (0,1)$ and $X_1, X_2, \ldots, Y_1, Y_2, \ldots$ be i.i.d. Bernoulli(p) r.v. We define

$$N := \min\{n \in \mathbb{N} \colon X_n \neq Y_n\}$$

and set "$Z := X_N$", *i.e*

$$Z = \sum_{n=1}^{\infty} \mathbb{1}_{\{N=n\}} X_n.$$

1. Check that $N \geqslant 1$ has the geometric distribution with parameter $2p(1-p)$.

2. Show that Z has the Bernoulli(1/2) distribution.

3. Deduce a way to simulate a fair coin toss using a potentially unfair coin.

Exercise 2.3.2. Let X be uniformly distributed on $[-1,1]$. Find the density of $Y := X^k$ for positive integers k.

Exercise 2.3.3. Let X have distribution function F. What is the distribution function of $Y := |X|$? When X admits a continuous density f_X, show that Y also admits a density f_Y, and express f_Y in terms of f_X.

Exercise 2.3.4. Let Θ be uniformly distributed on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$.

1. Find a continuous density function for $C := \tan\Theta$.

2. Find a density function for $A := (\sin\Theta)^2$ which is continuous on $(0,1)$.

3. Identify the law of $C^2 - AC^2 + A$.

Exercise 2.3.5. Let X be Cauchy with parameters $\alpha, 1$. Let $Y := a/X$ with $a \neq 0$. Show that Y is also a Cauchy r.v. and find its parameters.

Exercise 2.3.6. Let X, Y be two independent $\mathcal{N}(0,1)$ random variables. Find a density function for $Z := X^2/(X^2+Y^2)$ which is continuous on $(0,1)$.

Exercise 2.3.7. Let X be positive with a density f. Find a density for $Y := 1/(X+1)$.

Exercise 2.3.8. Let $X, X_1, X_2, \ldots$ be i.i.d. real r.v. with cumulative distribution function F and having a density function f. We set

$$N := \inf\{k \in \mathbb{N}\colon X_k > X\}.$$

1. Let $k \in \mathbb{N}$ and $t \in \mathbb{R}$. Show that

$$\mathbb{P}(N = k, X \leqslant t) = \int_{-\infty}^{t} F(x)^{k-1}\big(1 - F(x)\big) f(x)\,\mathrm{d}x.$$

2. Conclude that

$$\mathbb{P}(N = k) = \frac{1}{k} - \frac{1}{k+1}, \qquad k \in \mathbb{N}.$$

Exercise 2.3.9. Let X be a real random variable such that:

$$F_X(t) := \mathbb{P}(X \leqslant t) = \begin{cases} 0, & \text{if } t < -3, \\ 1/3, & \text{if } -3 \leqslant t < -2, \\ 7/12, & \text{if } -2 \leqslant t < 0, \\ 3/4, & \text{if } 0 \leqslant t < 4, \\ 1, & \text{if } 4 \leqslant t. \end{cases}$$

Compute $\mathbb{E}[X]$ and $\mathrm{Var}(X)$.

Exercise 2.3.10. Let $X, X_1, X_2, \ldots$ be i.i.d. real r.v. with distribution function F and having a density function f. We set

$$N := \inf\{k \in \mathbb{N}\colon X_k > X\}.$$

1. Let $k \in \mathbb{N}$ and $t \in \mathbb{R}$. Show that

$$\mathbb{P}(N = k, X \leqslant t) = \int_{-\infty}^{t} F(x)^{k-1}(1 - F(x)) f(x)\,\mathrm{d}x.$$

2. Conclude that

$$\mathbb{P}(N = k) = \frac{1}{k} - \frac{1}{k+1}, \qquad k \in \mathbb{N}.$$

Exercise 2.3.11. Let U, V be two independent standard uniform r.v. We set

$$X := U^2 + V^2, \quad \text{and} \quad Y := U^2/X.$$

Compute

$$\mathbb{P}(Y \leqslant t \mid X \leqslant 1) := \frac{\mathbb{P}(Y \leqslant t, X \leqslant 1)}{\mathbb{P}(X \leqslant 1)}, \qquad t \in \mathbb{R}.$$

Exercise 2.3.12. Let X be a real r.v. in $\mathrm{L}^1(\Omega, \mathscr{A}, \mathbb{P})$.

1. Let a, b be two real numbers. Show that

$$\mathbb{E}[|X-b|] - \mathbb{E}[|X-a|] = \int_a^b \big[\mathbb{P}(X \leqslant t) - \mathbb{P}(X \geqslant t)\big]\,\mathrm{d}t.$$

Hint. Observe that $|b-x| - |x-a| = \int_a^b \big(\mathbb{1}_{\{x \leqslant t\}} - \mathbb{1}_{\{x \geqslant t\}}\big)\,\mathrm{d}t$. Use Fubini's theorem.

2. We call $m \in \mathbb{R}$ a *median* of a real r.v. Y if $\mathbb{P}(Y \leqslant m) \geqslant 1/2$ and $\mathbb{P}(Y \geqslant m) \geqslant 1/2$.

   a) Show that every real random variable admits a median. Is there uniqueness?

   b) Let m be a median of X. Deduce from Question 1 that

   $$\mathbb{E}[|X-m|] = \inf_{c \in \mathbb{R}} \mathbb{E}[|X-c|].$$

   Conclude that $|\mathbb{E}[X] - m| \leqslant \sigma$ where $\sigma^2 := \mathrm{Var}(X)$.

Exercise 2.3.13. For any distribution function F, we define

$$F^{-1}(u) := \inf\{t \in \mathbb{R} \colon F(t) > u\}, \qquad u \in (0,1),$$

the *right-continuous inverse* of F.

1. Compute F^{-1} when F is the standard exponential distribution.

2. Show that for every $t \in \mathbb{R}$ and $u \in (0,1)$, $u < F(t) \implies F^{-1}(u) \leqslant t \implies u \leqslant F(t)$.

3. Let U be uniformly distributed on $(0,1)$.

   a) Show that $\lfloor \log_{1/2} U \rfloor$ has the Geometric$(1/2)$ distribution (with $\lfloor \cdot \rfloor$ = integer part).

   b) More generally, show that $F^{-1}(U)$ has law F.

4. Show that F^{-1} is non-decreasing.

5. Show that F^{-1} is right-continuous.

Consequently, the set $(0,1) \setminus \mathscr{C}(F^{-1})$ of discontinuity points of F^{-1} is at most countable.

6. Let $F, F_1, F_2, \dots$ be distribution functions such that $\forall t \in \mathscr{C}(F)$, $F_n(t) \to F(t)$. Show that $\forall u \in \mathscr{C}(F^{-1})$, $F_n^{-1}(u) \to F^{-1}(u)$.

7. Consider a convergence in distribution $X_n \Longrightarrow X$ of real r.v., and let U be a standard uniform r.v. Show that there exist Y and Y_n, $n \in \mathbb{N}$, measurable w.r.t. U such that $Y \sim X$, $Y_n \sim X_n$, and $Y_n \to Y$ a.s.

Exercise 2.3.14. Recall that a r.v. X has a *continuous distribution* if $x \mapsto \mathbb{P}(X \leqslant x)$ is continuous.

1. Show that X has a continuous distribution if and only if $\mathbb{P}(X = x) = 0$ for all $x \in \mathbb{R}$.

2. Show that if X has a continuous distribution and Y is any random variable independent of X, then $X+Y$ has a continuous distribution.

3. Let $f:\mathbb{R}\to[0,\infty)$ measurable. We suppose that (the distribution of) X has density f, that is

$$\mathbb{P}(X\in A)=\int_A f(x)\,\mathrm{d}x$$

for every Borel set A. Show that:

   a) f is integrable on $\mathbb{R}$.
   b) If $\mathbb{P}(X\in A)>0$, then A has positive Lebesgue measure.
   c) X has a continuous distribution.
   d) If X has another density g, then $f=g$ almost everywhere.

Exercise 2.3.15. Let L be the uniform distribution on $E:=(0,1)$, and $\mathbb{P}$ be the Arcsine distribution:

$$\mathbb{P}((0,t]) =: F(t)=\frac{1}{2}+\frac{\arcsin(2t-1)}{\pi},\qquad t\in E.$$

Define $X(s,t):=t\mathbb{1}_{\{s\leqslant t\}}+(1-t)\mathbb{1}_{\{s>t\}}$ for $s,t\in E$ and write $\mathbb{Q}$ for the law of X under $L\otimes\mathbb{P}$.

1. Show that for every bounded, measurable function $f\colon E\to\mathbb{R}$,

$$\int_E f(t)\,\mathbb{Q}(\mathrm{d}t)=\int_E 2tf(t)\,\mathbb{P}(\mathrm{d}t).$$

Deduce that $\mathbb{Q}$ admits w.r.t. $\mathbb{P}$ the Radon-Nikodym derivative

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}=2t,\qquad t\in E.$$

2. Conclude that X has density

$$t\mapsto\frac{2t}{\pi\sqrt{t(1-t)}},\qquad t\in E.$$

2.4 Convergence of random variables, limit theorems

Exercise 2.4.1. Let L^0 denote the space of real r.v. defined on $(\Omega,\mathbb{P})$.

1. Show that

$$d(X,Y):=\mathbb{E}[1\wedge|X-Y|]$$

is a distance on L^0 such that

$$X_n\xrightarrow[n\to\infty]{\mathbb{P}}X\iff d(X_n,X)\xrightarrow[n\to\infty]{}0.$$

2. Let $(X_{n,k}\colon n,k \geqslant 1)$ be elements in L^0, and $K\colon \Omega \to \mathbb{N}$ be an *independent* r.v. We suppose that for each $k \in \mathbb{N}$,

$$X_{n,k} \xrightarrow[n\to\infty]{\mathbb{P}} 0.$$

Show that

$$\sum_{k=1}^{K} X_{n,k} \xrightarrow[n\to\infty]{\mathbb{P}} 0.$$

Exercise 2.4.2. We have seen in Exercise 2.4.1 that the convergence in probability in the space $\mathrm{L}^0(\Omega,\mathscr{A},\mathbb{P})$ of real r.v. is metrized by

$$d(X,Y) := \mathbb{E}[1 \wedge |X-Y|].$$

0. Let $(X_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathrm{L}^0(\Omega,\mathscr{A},\mathbb{P})$:

$$\forall \varepsilon > 0,\ \exists k \in \mathbb{N},\ \forall m \geqslant k, \quad d(X_m, X_k) \leqslant \varepsilon.$$

a) Construct an increasing sequence $(k_n)_{n\geqslant 0}$ of positive integers such that

$$\mathbb{P}\left(\left|X_{k_{n+1}} - X_{k_n}\right| \geqslant \frac{1}{2^n}\right) \leqslant \frac{1}{2^n}.$$

b) Show that almost surely, there exists $N \geqslant 0$ sufficiently large such that

$$\forall n \geqslant N, \quad \left|X_{k_{n+1}} - X_{k_n}\right| \leqslant \frac{1}{2^n}.$$

Deduce that the sequence $(X_{k_n})_{n\geqslant 0}$ converges almost surely.

1. Prove that the space $\mathrm{L}^0(\Omega,\mathscr{A},\mathbb{P})$ is complete.

2. Prove that the space $\mathrm{L}^p(\Omega,\mathscr{A},\mathbb{P})$, $p \geqslant 1$, is complete.

Exercise 2.4.3. For each $p \in (0,1)$, let $B_k^{(p)}$, $k \in \mathbb{N}$, be i.i.d. Bernoulli(p) r.v. We set

$$X^{(p)} := \lim_{n\to\infty} X_n^{(p)}, \quad \text{where } X_n^{(p)} := \sum_{k=1}^{n} B_k^{(p)} 2^{-k},$$

and

$$A^{(p)} := \left\{ \sum_{k=1}^{\infty} b_k 2^{-k} \,\middle|\, b_k \in \{0,1\}, \text{ and } \lim_{k\to\infty} \frac{b_1 + \cdots + b_k}{k} = p \right\} \subset (0,1).$$

1. Show that $X^{(p)} \in A^{(p)}$ almost surely.

2. Show that for every $k,n \in \mathbb{N}_0$, $\mathbb{P}(k \leqslant 2^n X^{(p)} < k+1) \leqslant \theta^n$, with $\theta := \max(p, 1-p)$. Deduce that $X^{(p)}$ has a continuous distribution. We denote it $\mu^{(p)}$.

3. In this question we consider $p = 1/2$.

   a) Let U be a standard uniform r.v. Compute the characteristic function Φ_U.

   b) Show that for every $t \in \mathbb{R}$,

   $$\Phi_{X_n}(t) = \exp\big(it/2 - i2^{-(n+1)}t\big)\,\frac{\sin(t/2)}{2^n \sin\big(2^{-(n+1)}t\big)},$$

   and deduce that $\mu^{(1/2)}$ is the standard uniform distribution.
   Hint. Use that $(1+e^{i\theta})\sin(\theta/2) = e^{i\theta/2}\sin\theta$ to obtain a telescopic product.

4. We now consider $p \neq 1/2$.

   a) Show that $\mu^{(p)}(A^{(p)}) = 1$ and $\mu^{(1/2)}(A^{(p)}) = 0$.

   b) Deduce that $\mu^{(p)}$ has no density function.

Exercise 2.4.4. Let μ be a probability distribution on $\mathbb{R}$ having a second moment $\sigma^2 < \infty$ such that, if X and Y are independent with law μ, then the law of $(X+Y)/\sqrt{2}$ is also μ. Show that $\mu = \mathcal{N}(0,\sigma^2)$.
Hint. Apply the central limit theorem to packs of 2^n variables.

Exercise 2.4.5. Let X_n, $n \in \mathbb{N}$, be i.i.d. standard Poisson r.v., and $S_n := X_1 + \cdots + X_n$.
Find the expression of $\mathbb{P}\Big(\frac{S_n - n}{\sqrt{n}} \leqslant 0\Big)$, and deduce that $\lim\limits_{n\to\infty} e^{-n}\sum_{k=0}^{n}\frac{n^k}{k!} = \frac{1}{2}$.

Exercise 2.4.6. Let $X_1, X_2, \dots$ be i.i.d. real r.v. with $\mathrm{Var}(X_1) = 1$, $\mathbb{E}[X_1] = 0$, and

$$S_n := X_1 + \cdots + X_n, \qquad n \in \mathbb{N}.$$

1. Using the central limit theorem, show that there exist $p > 0$ and $n_0 \in \mathbb{N}$ such that

   $$\forall n \geqslant n_0, \quad \mathbb{P}(|S_n| \geqslant \sqrt{n}) \geqslant p.$$

2. Deduce that $\lim\limits_{n\to\infty} \mathbb{E}[|S_n|] = \infty$.

Exercise 2.4.7. For each $n \in \mathbb{N}$, let X_n be a $\mathcal{N}(\mu_n, \sigma_n^2)$ r.v. ($\mu_n \in \mathbb{R}$, $\sigma_n^2 > 0$). We suppose that X_n converges in distribution to some r.v. X.

1. Using characteristic functions, show that $(\sigma_n^2)_{n\in\mathbb{N}}$ converges to some $\sigma^2 \geqslant 0$.

2. Let $S(t) := \mathbb{P}(X > t)$, $t \in \mathbb{R}$.

   a) Justify that S is continuous at some $t_0 > 0$ large enough, with $S(t_0) < 1/4$.

   b) Deduce that $(\mu_n)_{n\in\mathbb{N}}$ is bounded from above (more precisely, $\limsup\limits_{n\to\infty} \mu_n \leqslant t_0$).

   c) Deduce that $(\mu_n)_{n\in\mathbb{N}}$ is bounded.

3. Conclude that X has a normal distribution.

Exercise 2.4.8. We suppose that $X, X_1, X_2, \dots$ are real r.v. such that X_n converges to X in distribution.

1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function. Show that

$$\liminf_{n\to\infty} \mathbb{E}[f(X_n)] \geqslant \mathbb{E}[f(X)].$$

   Hint. Apply Fatou's lemma/monotone convergence theorem to some $(f_k(X))_{k\in\mathbb{N}}$.

2. Deduce that if $(\mathbb{E}[|X_n|])_{n\in\mathbb{N}}$ is bounded, then $\mathbb{E}[|X|] < \infty$.
3. Deduce that if $X_n \geqslant 0$ a.s. for every $n \in \mathbb{N}$, then $X \geqslant 0$ a.s.

Exercise 2.4.9. Let $X_1, X_2, \dots$ be i.i.d. centered, square-integrable r.v. Show that

$$\liminf_{n\to\infty} \mathbb{P}(|X_1 + \cdots + X_n| \geqslant \sqrt{n}) > 0.$$

Exercise 2.4.10. Let $\lambda > 0$, and for $n > \lambda$, X_n be a random variable having the binomial distribution with parameter $(n, \lambda/n)$, that is

$$\mathbb{P}(X_n = k) = \binom{n}{k}\left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}, \qquad k = 0, 1, 2, \dots, n.$$

Compute $\lim_{n\to\infty} \mathbb{P}(X_n = k)$. What do you recognize?

Exercise 2.4.11. Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function, $x \in [0,1]$ and $X_n := X_n(x)$ be a random variable having the binomial distribution with parameter (n, x); see Exercise 2.1.4. We define $Y_n := f(X_n/n)$, so that Y_n is a discrete random variable taking values in the set $\mathscr{Y} := \{f(k/n)\colon k = 0, 1, 2, \dots, n\}$.

1. Let $m \in \mathbb{N}$. Recall why there exist $C > 0$ and $\delta_m > 0$ such that

$$\forall t \in [0,1], \quad |f(t)| \leqslant C,$$

   and

$$\forall (s,t) \in [0,1]^2 \text{ with } |t-s| \leqslant \delta_m, \quad |f(t) - f(s)| \leqslant \frac{1}{m}.$$

2. Check that for every $\delta > 0$,

$$\mathbb{E}[|Y_n - f(x)|] \leqslant 2C\,\mathbb{P}(|X_n - \mathbb{E}[X_n]| > n\delta) + \mathbb{E}[|f(X_n/n) - f(x)|\mathbb{1}_{\{|X_n - nx| \leqslant n\delta\}}],$$

   then deduce that for every $m \in \mathbb{N}$,

$$\mathbb{E}[|Y_n - f(x)|] \leqslant 2C\,\frac{x(1-x)}{n\delta_m^2} + \frac{1}{m}.$$

   Hint. Recall $\mathbb{E}[X_n]$, $\mathrm{Var}(X_n)$ (see Exercise 2.1.4), and apply Chebyshev's inequality.

3. We define $B_n\colon x \mapsto B_n(x) := \mathbb{E}[Y_n] = \mathbb{E}[f(X_n(x)/n)]$.

a) Check that B_n is a polynomial function in $x \in [0,1]$.

b) Conclude that

$$\sup_{x\in[0,1]} |B_n(x) - f(x)| \xrightarrow[n\to\infty]{} 0.$$

Conclusion. Continuous functions defined on a compact interval can be (uniformly) approximated by polynomials!

Exercise 2.4.12. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. r.v. with $\mathbb{E}[|X_1|] < \infty$, $\mu := \mathbb{E}[X_1]$, and

$$S_n := X_1 + \cdots + X_n, \qquad n \in \mathbb{N}.$$

1. Show that if $\mu > 0$ (resp. $\mu < 0$), then $S_n \longrightarrow \infty$ (resp. $-\infty$) almost surely.
2. We suppose here that $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2$.

   a) Let $m \geqslant 2k+1$ in $\mathbb{N}$. Show that $S_{n+m} - S_n = m$ infinitely often, almost surely. Deduce that $\limsup |S_n| > k$ almost surely.

   b) Conclude that $\limsup |S_n| = \infty$ a.s.

Exercise 2.4.13. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. real r.v. with distribution function F such that $F(t)/t \longrightarrow \lambda$ as $t \to 0^+$, for some $\lambda > 0$. Let $Z_n := n\min(X_1,\dots,X_n)$, $n \in \mathbb{N}$.

1. Check the following facts:

   a) For every $n \in \mathbb{N}$, $Z_n > 0$ almost surely.

   b) For every $t > 0$, $\mathbb{P}(Z_n > t) \longrightarrow e^{-\lambda t}$ as $n \to \infty$.

   c) For every $\varepsilon > 0$, there is $nX_n \leqslant \varepsilon$ infinitely often, almost surely.
2. Conclude that $\liminf Z_n = 0$ a.s., but that $(Z_n)_{n\in\mathbb{N}}$ does not converge a.s.

Exercise 2.4.14. For each $k \in \mathbb{N}$, let $(X_n^{(k)})_{n\in\mathbb{N}}$ be a sequence of real r.v. converging to 0 in probability, as $n \to \infty$. Define, for $k, n \in \mathbb{N}$,

$$Y_n^{(k)} := \sum_{i=1}^{k} X_n^{(i)},$$

and, for $\varepsilon > 0$ arbitrary,

$$f_n(k) := \mathbb{P}\left(\left|Y_n^{(k)}\right| > \varepsilon\right).$$

1. Let $k \in \mathbb{N}$. Show that $f_n(k) \longrightarrow 0$ ($Y_n^{(k)}$ converges to 0 in probability), as $n \to \infty$.
2. Let K be a $\mathbb{N}$-valued r.v. *independent* of $(X_n^{(k)})$, and $Y_n^{(K)}(\omega) := Y_n^{(K(\omega))}(\omega)$, $\omega \in \Omega$.

   a) Show that $\mathbb{P}(|Y_n^{(K)}| > \varepsilon) = \mathbb{E}[f_n(K)]$.

   b) Conclude that $Y_n^{(K)}$ converges to 0 in probability, as $n \to \infty$.

Exercise 2.4.15. Let X_n, $n \geqslant 1$, be centered with variance σ_n^2, such that $\sigma_n^2 \to 0$ as $n \to \infty$. Show that X_n converges to 0 in $\mathrm{L}^2(\mathbb{P})$ (and in probability).

Exercise 2.4.16. Let X_n, $n \geqslant 1$, be i.i.d. centered random variables with variance $\sigma^2 < \infty$. Show that $\frac{1}{n}\sum_{j=1}^n X_j$ converges to 0 in $\mathrm{L}^2(\mathbb{P})$ (and in probability).

Exercise 2.4.17. Let X_j, $j \geqslant 1$, be i.i.d. with standard Laplace distribution (having common density $e^{-|x|}/2$). Show the convergence in distribution

$$\sqrt{n}\,\frac{\sum_{j=1}^n X_j}{\sum_{j=1}^n X_j^2} \xrightarrow[n\to\infty]{D} Y,$$

where Y is a $\mathcal{N}(0,1/2)$ Gaussian variable.
Hint. Use Slutsky's lemma.

Exercise 2.4.18. Let X_j, $j \geqslant 1$, be i.i.d. with mean 1 and variance $\sigma^2 \in (0,\infty)$. Define $S_n := \sum_{j=1}^n X_j$, $n \in \mathbb{N}$. Show the convergence in distribution

$$\frac{2}{\sigma}(\sqrt{S_n} - \sqrt{n}) \xrightarrow[n\to\infty]{D} Y,$$

where Y is a $\mathcal{N}(0,1)$ Gaussian variable.

Exercise 2.4.19. Let X_j, $j \geqslant 1$, be i.i.d. with mean 0 and variance $\sigma^2 \in (0,\infty)$. Define $S_n := \sum_{j=1}^n X_j$, $n \in \mathbb{N}$. Show that $S_n/\sigma\sqrt{n}$ does not converge in probability.

Exercise 2.4.20. Let X_j, $j \geqslant 1$, be i.i.d. with mean 0 and variance $\sigma^2 < \infty$. Define $S_n := \sum_{j=1}^n X_j$, $n \in \mathbb{N}$. Show that

$$\lim_{n\to\infty} \mathbb{E}\left[\frac{|S_n|}{\sqrt{n}}\right] = \sqrt{\frac{2}{\pi}}\,\sigma.$$

Exercise 2.4.21. Let $q \geqslant 1$ and $(X_n)_{n\in\mathbb{N}}$ be a sequence of real r.v. bounded in $\mathrm{L}^q(\mathbb{P})$:

$$C := \sup_{n\in\mathbb{N}} \mathbb{E}[|X_n|^q] < \infty.$$

1. Suppose that X_n converges almost surely to some r.v. X as $n \to \infty$.
   a) Is X in $\mathrm{L}^q(\mathbb{P})$?
   b) Suppose $q > 1$ and $1 \leqslant p < q$. Does $\mathbb{E}[|X_n|^p]$ converge to $\mathbb{E}[|X|^p]$ as $n \to \infty$?
2. Same questions if the convergence $X_n \to X$ holds in probability.
3. Same questions if the convergence $X_n \to X$ holds in distribution.

Exercise 2.4.22. Let $X, X_1, \ldots$ be random variables and $g\colon \mathbb{R} \to [0,\infty)$ measurable. We suppose that

$$X_n \xrightarrow[n\to\infty]{(d)} X, \qquad \text{and} \quad \Theta := \sup_{n\geqslant 1} \mathbb{E}[g(X_n)] < \infty.$$

Show that for every continuous function $f\colon \mathbb{R} \to \mathbb{R}$ with $f = o(g)$ at $\pm\infty$, we have

$$\lim_{n\to\infty} \mathbb{E}[f(X_n)] = \mathbb{E}[f(X)] \text{ in } \mathbb{R}.$$

Exercise 2.4.23 (True or false?). Prove, or disprove (by giving a counterexample), briefly the following statements. We consider real r.v. on some general probability space $(\Omega, \mathscr{A}, \mathbb{P})$.

1. If $|X_n - X| \to 0$ a.s., then $\mathbb{E}[|X_n - X|] \to 0$.
2. If $X_n \to X$ in probability and $(\mathbb{E}[X_n^2])_{n\in\mathbb{N}}$ is bounded, then $\mathbb{E}[|X_n - X|] \to 0$.
3. If X_n tends to 0 in probability, then so does $(X_1 + \cdots + X_n)/n$.
4. If $\mathbb{E}[|X_n - X|] \to 0$, then $|X_n - X| \to 0$ a.s.

2.5 Gaussian vectors

Exercise 2.5.1. Let $\mathbf{X} := (X_1, X_2, X_3) \in \mathbb{R}^3$ be a centered random Gaussian vector such that $\mathbb{E}[X_i^2] = 1$ and $\mathbb{E}[X_i X_j] = 1/2$ for $1 \leqslant i \neq j \leqslant 3$.

1. Give the dispersion matrix and the characteristic function of $\mathbf{X}$.
2. What is the law of $X_1 - X_2 + 2X_3$?
3. Does there exist $a \in \mathbb{R}$ such that $X_1 + aX_2$ and $X_1 - X_2$ are independent?
4. Show that $\mathbf{X}$ admits a density and explicit it.

Exercise 2.5.2. Let $a > 0$, X be a $\mathcal{N}(0,1)$ random variable, and

$$Y := \begin{cases} X, & \text{if } |X| < a, \\ -X, & \text{if } |X| \geqslant a. \end{cases}$$

1. Show that Y has the $\mathcal{N}(0,1)$ distribution.
2. Express $\mathbb{E}[XY]$ in terms of the density function $f(x) := \exp(-x^2/2)/\sqrt{2\pi}$ of X.
3. Is (X, Y) a Gaussian random vector?

Exercise 2.5.3. Let $n \geqslant 2$ and $X_1, \dots, X_n$ be i.i.d. $\mathcal{N}(\mu, \sigma^2)$ r.v. Prove that the empirical mean and variance

$$\bar{X}_n := \frac{1}{n}\sum_{i=1}^n X_i \quad \text{and} \quad S_n^2 := \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X}_n)^2$$

are independent.

Hint. Let $\mathbf{X}' := (X_1 - \bar{X}_n, \dots, X_n - \bar{X}_n)$. Check that $\mathbf{X} := (\bar{X}_n, \mathbf{X}') \in \mathbb{R}^{n+1}$ is a Gaussian vector. Express its dispersion matrix using the one of $\mathbf{X}'$ and deduce that $\bar{X}_n$ and $\mathbf{X}'$ are independent.

Exercise 2.5.4. Let $\mathbf{X}_1, \mathbf{X}_2, \dots$ be i.i.d. random vectors in $\mathbb{R}^2$. Apply the 2-dimensional CLT in the following cases:

1. $\mathbb{P}(\mathbf{X}_1 = (-1,-1)) = \mathbb{P}(\mathbf{X}_1 = (1,1)) = 1/2$;
2. $\mathbb{P}(\mathbf{X}_1 = (1,-1)) = \mathbb{P}(\mathbf{X}_1 = (1,1)) = \mathbb{P}(\mathbf{X}_1 = (-1,-1))/2 = 1/4$.

Exercise 2.5.5. Let $\mathbf{X}_1, \mathbf{X}_2, \dots$ be i.i.d. vectors in $\mathbb{R}^k$ having the same distribution as $\mathbf{X} := (\xi_1, \xi_1 + \xi_2, \dots, \xi_1 + \dots + \xi_k)$, for $\xi_1, \dots, \xi_k$ i.i.d. with $\mathbb{P}(\xi_1 = 1) = \mathbb{P}(\xi_1 = -1) = 1/2$. Show that $(\mathbf{X}_1 + \dots + \mathbf{X}_n)/\sqrt{n}$ has a limiting distribution which one will describe in terms of a density function.

Exercise 2.5.6. Let ρ be in between -1 and 1, and μ_j, σ_j^2, $j = 1, 2$, be given. Construct Gaussian variables X_1, X_2 with means μ_1, μ_2, variances σ_1^2, σ_2^2, and correlation ρ.

Exercise 2.5.7. Let (X, Y) be bivariate normal with correlation ρ and $\sigma_X^2 = \sigma_Y^2$. Show that X and $Y - \rho X$ are independent.

Exercise 2.5.8. Let $\mathbf{X} := (X_1, X_2, \dots, X_n)$ be a n-dimensional centered Gaussian vector. We suppose that there exist $k \geqslant 2$ and $0 = i_0 < \dots < i_k = n$ such that the covariance matrix Q of $\mathbf{X}$ is a block-diagonal matrix consisting of k blocks $Q_1, \dots, Q_k$, *i.e*,

$$Q = \begin{pmatrix} Q_1 & & (0) \\ & \ddots & \\ (0) & & Q_k \end{pmatrix},$$

with respective sizes $i_1 - i_0, \dots, i_k - i_{k-1}$. Show that $\mathbf{X}_j := (X_{i_{j-1}+1}, \dots, X_{i_j})$, $1 \leqslant j \leqslant k$, are independent centered Gaussian vectors with respective covariance matrices Q_j.

Exercise 2.5.9. Let $\mathbf{X}$ be Gaussian $\mathcal{N}(\boldsymbol{\mu}, Q)$ in $\mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, $\mathbf{b} \in \mathbb{R}^n$, and $\mathbf{Y} := A\mathbf{X} + \mathbf{b}$.

1. Show that $\mathbf{Y}$ is still a Gaussian vector. Give its parameters in terms of $Q, A, \boldsymbol{\mu}, \mathbf{b}$.

2. Show that $\mathbf{Y}$ is nondegenerate if and only if $\mathbf{X}$ is nondegenerate and A is invertible.

3. We suppose $\det(Q) \neq 0$. Find A and $\mathbf{b}$ such that $\mathbf{Y}$ is standard $\mathcal{N}(\mathbf{0}, I)$.

Exercise 2.5.10. Let $\mathbf{Y} := (Y_1, \dots, Y_n)$ be a nondegenerate Gaussian vector with covariance matrix Q, X be some random variable with finite variance, and $\mathbf{v} := (v_1, \dots, v_n) \in \mathbb{R}^n$. Show that

$$\operatorname{Var}\left(\sum_{i=1} v_i Y_i - X\right)$$

is minimal for $\mathbf{v} = Q^{-1}\mathbf{u}$, where $\mathbf{u} := (u_1, \dots, u_n)$ is given by $u_i = \operatorname{Cov}(Y_i, X)$, $1 \leqslant i \leqslant n$.

Exercise 2.5.11. Let (X, Y) be a nondegenerate 2-dimensional centered Gaussian vector, and

$$Z := \begin{cases} X, & \text{if } X^2 + Y^2 < 1, \\ -X, & \text{else.} \end{cases}$$

Show that Z is Gaussian, $Z \sim X$, but that (X, Y, Z) is not a Gaussian vector.

2.6 Conditional expectations

Exercise 2.6.1. Let X,Y be two independent Poisson variables with parameters $\lambda,\mu>0$ respectively. We set $N := X+Y$.

1. Compute $\mathbb{P}(X=k \mid N=n)$ for $k,n\in\mathbb{Z}_+$.
2. Deduce $\mathbb{E}[X \mid N=n]$ for $n\in\mathbb{Z}_+$, and then $\mathbb{E}[X\mid N]$.
3. Check that $\mathbb{E}[X]=\mathbb{E}[\mathbb{E}[X\mid N]]$.

Exercise 2.6.2. Let X,Y be two independent exponential r.v. with parameters $\lambda,\mu>0$ respectively. We set $T:=\min(X,Y)$.

1. What is the law of T?
2. Compute $\mathbb{E}[T\mid X]$.

   Hint. Go back to definitions. (Find $\mathbb{E}[Tf(X)]$ for $f:\mathbb{R}\to\mathbb{R}$ measurable bounded...)
3. Compute $\mathbb{E}[X\mid T]$.
4. Check that $\mathbb{E}[\mathbb{E}[T\mid X]]=\mathbb{E}[T]$ and $\mathbb{E}[\mathbb{E}[X\mid T]]=\mathbb{E}[X]$.

Exercise 2.6.3. Let U,V be two independent standard uniform r.v. on $(0,1)$. Compute

$$\mathbb{E}[(U-V)^+ \mid U].$$

Exercise 2.6.4. Let $X,Y\in \mathrm{L}^1(\Omega,\mathscr{A},\mathbb{P})$.

1. Show that if $X=Y$ a.s., then $\mathbb{E}[X\mid Y]=\mathbb{E}[Y\mid X]$ a.s.
2. Conversely, show that if $\mathbb{E}[X\mid Y]=Y$ and $\mathbb{E}[Y\mid X]=X$ a.s., then $X=Y$ a.s.

   Hint. You may only consider the case $X,Y\in\mathrm{L}^2(\mathbb{P})$ (show that $\mathbb{E}[(X-Y)^2]=0$).

Exercise 2.6.5. Let $\mathbf{X}:=(X_1,\dots,X_d)$ be a $\mathscr{N}(0,\Gamma)$ centered Gaussian vector in $\mathbb{R}^d$. Compute $\mathbb{E}[\langle\boldsymbol{\lambda},\mathbf{X}\rangle \mid \langle\boldsymbol{\mu},\mathbf{X}\rangle]$ for $\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathbb{R}^d$ (with $\langle\cdot,\cdot\rangle$ the usual inner product in $\mathbb{R}^d$).

Exercise 2.6.6. Suppose $(B_n)\in\mathscr{A}^{\mathbb{N}}$ is a partition of Ω (that is, $\Omega=\bigcup_{n\geqslant 1}B_n$ with $B_n\neq\emptyset$ and $B_n\cap B_m=\emptyset$ for $n\neq m$), and let $\mathscr{B}:=\sigma(B_n:\ n\geqslant 1)$. Show that for every $X\in\mathrm{L}^1(\Omega,\mathscr{A},\mathbb{P})$,

$$\mathbb{E}[X\mid\mathscr{B}] = \sum_{n=1}^{\infty}\mathbb{E}[X\mid B_n]\,\mathbb{1}_{B_n}.$$

Exercise 2.6.7. Let $(\Omega,\mathscr{A},\mathbb{P})$ be a probability space, $\mathscr{B}\subseteq\mathscr{A}$ be a sub-σ-field, and $A\in\mathscr{A}$ be an event. Show that the event $B:=\{\mathbb{P}(A\mid\mathscr{B})>0\}$ contains a.s. A (that is, $\mathbb{P}(A\setminus B)=0$).

Exercise 2.6.8. Let $X \in \mathrm{L}^2(\Omega, \mathscr{A}, \mathbb{P})$ and $\mathscr{B} \subseteq \mathscr{A}$ a sub-σ-field. We define the conditional variance of X w.r.t. $\mathscr{B}$ by:

$$\operatorname{Var}(X \mid \mathscr{B}) := \mathbb{E}\big[(X - \mathbb{E}[X \mid \mathscr{B}])^2 \,\big|\, \mathscr{B}\big].$$

Prove the *law of total variance*:

$$\operatorname{Var}(X) = \mathbb{E}\big[\operatorname{Var}(X \mid \mathscr{B})\big] + \operatorname{Var}\big(\mathbb{E}[X \mid \mathscr{B}]\big).$$

Exercise 2.6.9. Let $X_1, X_2, \ldots$ be i.i.d. r.v. in $\mathrm{L}^1(\mathbb{P})$, and $S_n := X_1 + \cdots + X_n$, $n \geqslant 1$.

1. Find $\mathbb{E}[X_1 \mid X_2]$, $\mathbb{E}[S_n \mid X_1]$, and $\mathbb{E}[S_n \mid S_{n-1}]$.

2. Show that if (X, Z) and (Y, Z) have the same joint law, then for every $f : \mathbb{R} \to \mathbb{R}$ with $f(X) \in \mathrm{L}^1(\mathbb{P})$, we have $\mathbb{E}[f(X) \mid Z] = \mathbb{E}[f(Y) \mid Z]$. Deduce $\mathbb{E}[X_1 \mid S_n]$.

Exercise 2.6.10. Let $p \in (0,1]$, let X_n, $n \in \mathbb{N}$, be a Binomial(n, p) r.v., and, given X_n, let Y_n have a Poisson(X_n) distribution.

1. Compute the mean m_n, the variance σ_n^2, and the characteristic function Φ_n of Y_n.

2. Show that

$$\frac{Y_n - m_n}{\sigma_n} \xrightarrow[n\to\infty]{(d)} Z,$$

where $Z \sim \mathcal{N}(0,1)$. Is there a link with the central limit theorem?

Exercise 2.6.11. Let U be a uniformly distributed r.v. on $[0,1)$ and let $X_n := \lfloor nU \rfloor$ for $n \geqslant 1$. Determine the conditional law of U given X_n.

Exercise 2.6.12. Let (X, Y) be a random vector in $\mathbb{R}^{n+m}$ with probability density function (p.d.f.) p.

1. Show that $Y \in \mathbb{R}^m$ admits a p.d.f. q and give its expression in terms of p.

2. For each $y \in \mathbb{R}^m$, we let $\nu(y, \cdot)$ denote the measure on $\mathbb{R}^n$ given by

$$\nu(y, A) := \frac{1}{q(y)} \int_A p(x, y)\,\mathrm{d}x, \qquad A \in \mathscr{B}(\mathbb{R}^n)$$

(with the convention $\nu(y, A) = 0$ if $q(y) = 0$). Prove that for every bounded measurable function $f : \mathbb{R}^{n+m} \to \mathbb{R}$,

$$\mathbb{E}[f(X, Y) \mid Y] = \int f(x, Y)\,\nu(Y, \mathrm{d}x).$$

Exercise 2.6.13. Let $\{X_n\}_{n \geqslant 0} \subset \mathrm{L}^2(\mathbb{P})$ such that $S_n := X_1 + \cdots + X_n$, $n \geqslant 0$, defines a martingale. Show that $\mathbb{E}[X_i X_j] = 0$ for all $i \neq j$.

2.7 Martingales

Exercise 2.7.1. Let $(X_n)_{n\geqslant 0}$ be a martingale and T a stopping time. Recall that $(X_{n\wedge T})_{n\geqslant 0}$ is again a martingale and that (optional stopping theorem) if $\boldsymbol{T} \in \mathrm{L}^\infty$, then

$$X_T \in \mathrm{L}^1, \quad \text{with } \mathbb{E}[X_T] = \mathbb{E}[X_0]. \tag{$\star$}$$

Show that ($\star$) also holds in the other two following cases:

1. When $T < \infty$ a.s. and $(X_n)_{n\geqslant 0}$ is dominated by some r.v. in L^1.
2. When $T \in \mathrm{L}^1$ and $(X_{n+1} - X_n)_{n\geqslant 0}$ is bounded in L^∞.

Hint. Use the dominated convergence theorem.

Exercise 2.7.2 (Pig). Let D_i, $i \geqslant 1$, be i.i.d. realizations of a fair 6-faced die roll. We define

$$T := \inf\{i \geqslant 1 \colon D_i = 1\},$$
$$\mathscr{F}_n := \sigma(D_1, \ldots, D_n), \qquad n \geqslant 0,$$

and

$$S_n := \sum_{i=1}^{n} D_i, \qquad n \geqslant 0.$$

1. Check that T is a $(\mathscr{F}_n)_{n\geqslant 0}$-stopping time. Compute $\mathbb{E}[T]$.
2. Show that
$$\mathbb{E}[S_n \mid T] = 4n\,\mathbb{1}_{\{T>n\}} + \left(\frac{7n+T}{2} - 3\right)\mathbb{1}_{\{T\leqslant n\}}.$$
3. Deduce that $\mathbb{E}[S_T] = 21$.
4. Provide an alternative solution to Question 3 using a martingale.

   Hint. Determine $m \in \mathbb{R}$ such that $(S_n - mn)_{n\geqslant 0}$ is a $(\mathscr{F}_n)_{n\geqslant 0}$-martingale.

Exercise 2.7.3 (Pokémon Go). Imagine that at each time $n = 1, 2, \ldots$, you find one of the m Pokémon™, assuming they all appear independently and uniformly at random. Let $R_0 := m$ and R_n be the number of different Pokémon you still need to capture after time n in order to complete your Pokédex.

1. Justify that conditionally on R_n, the r.v. $R_n - R_{n+1}$ is Bernoulli(R_n/m) distributed.
2. Let $h(k) := \sum_{1\leqslant i\leqslant k} 1/i$ for every $k \geqslant 0$. Deduce from Question 1 that, respectively,
$$M_n := \left(\frac{m}{m-1}\right)^n R_n, \quad \text{and} \quad L_n := \frac{n}{m} + h(R_n), \qquad n \geqslant 0,$$
   define a martingale and a submartingale w.r.t. the natural filtration $(\mathscr{F}_n)_{n\geqslant 0}$.
3. Let $T := \inf\{n \geqslant 0 \colon R(n) = 0\}$ be the time you catch them all.
   a) Check that T is a $(\mathscr{F}_n)_{n\geqslant 0}$-stopping time.

b) Show that $X_n := L_{n\wedge T}$, $n \geqslant 0$, becomes a martingale and deduce that $T \in \mathrm{L}^1$.

c) Deduce $\mathbb{E}[T]$ (apply Exercise 2.7.1). Give an equivalent when m is large.

Exercise 2.7.4. Let $\theta \in \mathbb{R}$, $(X_k)_{k\in\mathbb{N}}$ be a sequence of i.i.d. $\mathcal{N}(0,1)$ r.v., and

$$S_n := \sum_{k=1}^{n} X_k, \qquad n \geqslant 0.$$

1. Find $f\colon \mathbb{R} \to \mathbb{R}$ such that $M_n^{(\theta)} := \exp(\theta S_n - n f(\theta))$, $n \geqslant 0$, is a martingale.

2. Does $M_n^{(\theta)}$ converge as $n \to \infty$, almost surely? in L^1?

Exercise 2.7.5. Let $E := \{\mathbf{A}, \mathbf{B}\}$ be a set with two elements, $m \in \mathbb{N}$, and consider an initial population $X_0 \in E^m$ of m individuals, each of which has either type **A** or type **B**. Suppose that at each time $n = 1, 2, \ldots$, a new population X_n is born in such a way that each individual inherits the type of one individual in the previous generation X_{n-1}, which is chosen independently and uniformly at random. Formally

$$X_n = (X_{n-1}(\sigma_{n,1}), \ldots, X_{n-1}(\sigma_{n,m})) \in E^m,$$

with $(\sigma_{n,i})_{n\in\mathbb{N}, 1\leqslant i\leqslant m}$ an independent family of i.i.d. uniform r.v. on $\{1, \ldots, m\}$.

1. What do you think will eventually happen to the population?

Let A_n, $n \geqslant 0$, denote the number of individuals in X_n which have type **A**.

2. Justify that conditionally on A_n, the r.v. A_{n+1} is Binomial$(m, A_n/m)$ distributed.

3. Show that $(A_n)_{n\geqslant 0}$ is a martingale converging a.s.

4. Check that $\mathbb{E}[A_{n+1}^2 \mid A_n] = \frac{m-1}{m} A_n^2 + A_n$. Deduce that

$$M_n := (m-1)(m - A_n) + (m A_n - A_n^2)\left(\frac{m}{m-1}\right)^n, \qquad n \geqslant 0,$$

defines another martingale.

5. Prove your conjecture in Question 1.

Exercise 2.7.6. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent r.v. We suppose that there exists a constant $C > 0$ such that the following three (deterministic) series

$$\text{(a)} \sum_{n\in\mathbb{N}} \mathbb{P}(|X_n| > C), \qquad \text{(b)} \sum_{n\in\mathbb{N}} \mathbb{E}\big[X_n \mathbb{1}_{\{|X_n|\leqslant C\}}\big], \qquad \text{(c)} \sum_{n\in\mathbb{N}} \mathrm{Var}\big(X_n \mathbb{1}_{\{|X_n|\leqslant C\}}\big),$$

all converge (in $\mathbb{R}$). Show that the series $\sum_{n\in\mathbb{N}} X_n$ converges almost surely.

Hint. Show that $M_n := \sum_{k=1}^{n} \big(X_k \mathbb{1}_{\{|X_k|\leqslant C\}} - \mathbb{E}\big[X_k \mathbb{1}_{\{|X_k|\leqslant C\}}\big]\big)$, $n \geqslant 0$, is bounded in $\mathrm{L}^2(\mathbb{P})$.

Exercise 2.7.7 (A counterexample). Let T be a r.v. in $\mathbb{N}$ and $(Y_k)_{k\in\mathbb{N}}$ be an independent family of i.i.d. r.v. with $\mathrm{Var}(Y_1) = 1$ and $\mathbb{E}[Y_1] = 0$. We set $\mathscr{F}_n := \sigma(T, Y_1, \ldots, Y_n)$ and

$$X_n := \sum_{k=1}^{n} Y_k, \qquad n \geqslant 0.$$

1. Show that $(X_n)_{n\geqslant 0}$ is a $(\mathscr{F}_n)_{n\geqslant 0}$-martingale which is not bounded in $\mathrm{L}^1(\mathbb{P})$.
2. Give an example of distribution for T such that the $(\mathscr{F}_n)_{n\geqslant 0}$-stopped martingale $(X_{n\wedge T})_{n\in\mathbb{N}}$ is still not bounded in $\mathrm{L}^1(\mathbb{P})$ (although it converges almost surely).

Exercise 2.7.8. Let S, T be two $(\mathscr{F}_n)_{n\geqslant 0}$-stopping times and $X \in \mathrm{L}^1(\mathbb{P})$. Show that

$$\mathbb{E}[\mathbb{E}[X \mid \mathscr{F}_S] \mid \mathscr{F}_T] = \mathbb{E}[\mathbb{E}[X \mid \mathscr{F}_T] \mid \mathscr{F}_S] = \mathbb{E}[X \mid \mathscr{F}_{S\wedge T}].$$

Hint. Apply the stopping theorem... a few times.

Exercise 2.7.9 (Other counterexamples). Let $f \colon \mathbb{N} \to \mathbb{R}$ measurable and T be a $\mathbb{N}$-valued r.v. such that $f(T) \in \mathrm{L}^1(\mathbb{P})$. For every $n \geqslant 0$, define $\mathscr{F}_n := \sigma(\{T = k\}, k \leqslant n)$ and

$$X_n := \mathbb{1}_{\{T\leqslant n\}} f(T) + \mathbb{1}_{\{T>n\}} r(n), \quad \text{where } \ r(n) := \frac{\mathbb{E}[\mathbb{1}_{\{T>n\}} f(T)]}{\mathbb{P}(T > n)}.$$

1. Check that T is a $(\mathscr{F}_n)_{n\geqslant 0}$-stopping time and that $(X_n)_{n\geqslant 0}$ is a uniformly integrable $(\mathscr{F}_n)_{n\geqslant 0}$-martingale.
2. In this question we suppose that $f(k) = 2^k k^{-2}$ and that $\mathbb{P}(T = k) = 2^{-k}$, $k \in \mathbb{N}$.
   a) Show that $X_{T-1} \notin \mathrm{L}^1(\mathbb{P})$. *What is wrong regarding the stopping theorem?*
   b) Deduce that $(X_n)_{n\geqslant 0}$ is not dominated in $\mathrm{L}^1(\mathbb{P})$.
3. In this question we suppose that $f(k) = \log k$, $k \in \mathbb{N}$, and that, as $k \to \infty$,
$$\mathbb{P}(T = k) = \frac{1}{k^2(\log k)^2} + O\biggl(\frac{1}{k^2(\log k)^3}\biggr).$$
   a) Check that $T \in \mathrm{L}^1(\mathbb{P})$, while $T \notin \mathrm{L}^2(\mathbb{P})$.
   b) Show that $(X_{n+1} - X_n)_{n\geqslant 0}$ is bounded in $\mathrm{L}^\infty(\mathbb{P})$.
   Hint. $\displaystyle\sum_{k>n} \frac{1}{k^2(\log k)^p} = \frac{1}{n(\log n)^p} + O\biggl(\frac{1}{n(\log n)^{p+1}}\biggr)$ as $n \to \infty$, for $p \in \{1, 2\}$.
   c) Show that $\displaystyle\sum_{k=1}^{T} X_k \notin \mathrm{L}^1(\mathbb{P})$.

Exercise 2.7.10. On the filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_n)_{n\geqslant 0}, \mathbb{P})$, let $(X_n)_{n\geqslant 0}$ be a martingale and T be a stopping time. We suppose that

$$\mathbb{P}(T < \infty) = 1, \quad \mathbb{E}[|X_T|] < \infty, \quad \text{and} \quad \lim_{n\to\infty} \mathbb{E}[|X_n| \mathbb{1}_{\{T>n\}}] = 0.$$

Show that $\mathbb{E}[X_T] = \mathbb{E}[X_0]$.

Exercise 2.7.11. On $(\Omega, \mathscr{F}, (\mathscr{F}_n)_{n\geqslant 0}, \mathbb{P})$, let $(X_n)_{n\geqslant 0}$ be an adapted, integrable process,

$$A_n := \sum_{k=1}^{n} \mathbb{E}[X_k - X_{k-1} \mid \mathscr{F}_{k-1}], \qquad n \geqslant 0,$$

and

$$M_n := X_n - A_n, \qquad n \geqslant 0.$$

1. Show that $(A_n)_{n\geqslant 0}$ is a predictable process.
2. Show that $(M_n)_{n\geqslant 0}$ is a martingale.
3. Suppose we are given a predictable process $(A'_n)_{n\geqslant 0}$ with $A'_0 = 0$ and a martingale $(M'_n)_{n\geqslant 0}$ such that $X_n = M'_n + A'_n$, $n \geqslant 0$. Show that $A'_n = A_n$ and $M'_n = M_n$ a.s. for all $n \geqslant 0$.
4. We suppose in this question that $(X_n)_{n\geqslant 0}$ is a nonnegative submartingale.
   a) Show that $A_n \leqslant A_{n+1}$ a.s. for all $n \geqslant 0$. We write $A_\infty := \lim_{n\to\infty} A_n \in [0,\infty]$.
   b) Show that if $\mathbb{E}[A_\infty] < \infty$, then $(X_n)_{n\geqslant 0}$ converges a.s.
   c) For $a > 0$, let $T_a := \inf\{n \geqslant 0 \colon A_{n+1} > a\}$.
      i- Check that T_a is a stopping time, and that $\mathbb{E}[X_{n\wedge T_a}] \leqslant a + \mathbb{E}[X_0]$.
      ii- Deduce that $(X_n)_{n\geqslant 0}$ converges a.s. on the event $\{T_a = \infty\}$.
      iii- Conclude that $(X_n)_{n\geqslant 0}$ converges a.s. on the event $\{A_\infty < \infty\}$.
   d) We suppose that the increments of $(X_n)_{n\geqslant 0}$ are dominated in $\mathrm{L}^1(\Omega, \mathscr{F}, \mathbb{P})$:
   $$\mathbb{E}[S] < \infty, \quad \text{where } S := \sup_{n\geqslant 1} |X_n - X_{n-1}|. \tag{$\star$}$$
   Show that $\limsup_{n\to\infty} X_n = \infty$ a.s. on the event $\{A_\infty = \infty\}$.
5. We suppose in this question that $(X_n)_{n\geqslant 0}$ is a martingale satisfying to ($\star$). Show that a.s. as $n \to \infty$, X_n either converges or oscillates, that is
$$\lim_{n\to\infty} X_n \text{ exists in } \mathbb{R} \quad \text{or} \quad \left(\liminf_{n\to\infty} X_n = -\infty \text{ and } \limsup_{n\to\infty} X_n = \infty\right).$$
6. Prove the *conditional Borel–Cantelli lemma*: if $E_n \in \mathscr{F}_n$, $n \in \mathbb{N}$, then (up to a $\mathbb{P}$-null set)
$$\{\limsup E_n\} = \left\{\sum_{n=1}^{\infty} \mathbb{P}(E_n \mid \mathscr{F}_{n-1}) = \infty\right\}.$$

Exercise 2.7.12. Let $U_1, U_2, \ldots$ be i.i.d. Uniform$(0,1)$ r.v. Let X_0 be any r.v. on $(0,1)$ independent of $(U_i)_{i\geqslant 1}$, and define by induction

$$X_n := tX_{n-1} + (1-t)\mathbb{1}_{\{U_n \leqslant X_{n-1}\}}, \qquad n \geqslant 1,$$

where $t \in (0,1)$ is fixed.

1. Show that $(X_n)_{n\geqslant 0}$ is a martingale converging a.s. and in L^p for every $p\geqslant 1$.

2. Determine the law of $X_\infty := \lim_{n\to\infty} X_n$.
   Hint. Compute $\mathbb{E}[(X_{n+1}-X_n)^2]$.

Exercise 2.7.13. Let X_n, $n\geqslant 1$, be independent nonnegative r.v. with mean 1, and

$$M_n := \prod_{i=1}^{n} X_i, \qquad n\geqslant 0.$$

1. Show that $M_\infty := \lim_{n\to\infty} M_n$ exists almost surely, and $\mathbb{E}[M_\infty]\leqslant 1$.

2. Let $a_n := \mathbb{E}[\sqrt{X_n}]$, $n\geqslant 1$. Prove that the following conditions are equivalent:

   (a) $\mathbb{E}[M_\infty]=1$;
   (b) $M_n\to M_\infty$ in $\mathrm{L}^1(\mathbb{P})$;
   (c) $(M_n)_{n\geqslant 0}$ is uniformly integrable;
   (d) $\prod_{k\geqslant 1} a_k > 0$;
   (e) $\sum_{k\geqslant 1}(1-a_k)<\infty$.

3. Show that if one of the above condition is not satisfied, then $M_\infty = 0$ a.s.

4. Express M_∞ in the particular case where the X_n, $n\in\mathbb{N}$, are i.i.d.

Exercise 2.7.14 (Counterexamples).

1. Let U be a Uniform$(0,1)$ r.v. and $X_n := n\mathbb{1}_{\{nU<1\}}$, $n\geqslant 1$. Show that $(X_n)_{n\geqslant 1}$ is bounded in $\mathrm{L}^1(\mathbb{P})$ but not uniformly integrable.

2. Show that the two following families are uniformly integrable but not dominated in $\mathrm{L}^1(\mathbb{P})$ (that is, $\mathbb{E}[\sup_{X\in\mathscr{X}}|X|]=\infty$):

   a) $\mathscr{X} := \{X_{n,k}\}_{\substack{n\geqslant 0\\ 0\leqslant k<2^{2n}}}$ with $X_{n,k} := 2^n\mathbb{1}_{\{k\leqslant 2^{2n}U<k+1\}}$ and $U\sim$ Uniform$(0,1)$;

   b) $\mathscr{X} := \{X_n\}_{n\geqslant 1}$ with $X_n := nA_nB_n$, A_n, B_n, $n\geqslant 1$, Bernoulli$(\frac{1}{n})$ r.v., all independent.
   Hint. Use Borel–Cantelli lemmas to prove that $X_n\to 0$ a.s., and that $\mathbb{E}[X_n\,|\,\mathscr{F}]\to 0$ in $\mathrm{L}^1(\mathbb{P})$ but not a.s., where $\mathscr{F} := \sigma(A_n\colon n\geqslant 1)$.

3. Let X_n, $n\geqslant 1$, be independent r.v. with $\mathbb{P}(X_n = 1-n^2) = 1-\mathbb{P}(X_n=1) = n^{-2}$. Show that $S_n := X_1+\cdots+X_n$, $n\geqslant 0$, defines a martingale converging a.s. to $+\infty$. Is this in contradiction with Exercise 2.7.11.5?

Exercise 2.7.15. Let $\mathscr{S} := \bigcup_{n\geqslant 1}\mathscr{S}_n$, where $\mathscr{S}_n := \{\pi\colon \mathbb{N}\to\mathbb{N}$ bijective with $\pi(k)=k$ for all $k>n\}$. Suppose $X := (X_n)_{n\geqslant 1}$ is a stochastic process such that for every $\pi\in\mathscr{S}$, $X^\pi := (X_{\pi(n)})_{n\geqslant 1}$ has the same law as X. Define the *exchangeable σ-algebra* $\mathscr{E} := \bigcap_{n\geqslant 1}\mathscr{E}_n$, where

$$\mathscr{E}_n := \Big\{\{X\in A\}\colon A\subseteq\mathbb{R}^{\mathbb{N}} \text{ measurable s.t. } \{X\in A\}=\{X^\pi\in A\} \text{ for all } \pi\in\mathscr{S}_n\Big\}.$$

1. Show that for every $f:\mathbb{R}^{\mathbb{N}}\to\mathbb{R}$ bounded measurable,

$$\mathbb{E}[f(X)\mid\mathscr{E}_n]=\frac{1}{n!}\sum_{\pi\in\mathscr{S}_n}f(X^\pi),\quad n\geqslant 1,$$

   and that as $n\to\infty$, this sequence converges to $\mathbb{E}[f(X)\mid\mathscr{E}]$ a.s. and in $\mathrm{L}^1(\mathbb{P})$.

2. Let the *tail σ-algebra* $\mathscr{T}:=\bigcap_{n\geqslant 1}\sigma(X_k\colon k\geqslant n)$. Show that $\mathscr{T}\subseteq\mathscr{E}$ and that for all $f:\mathbb{R}^{\mathbb{N}}\to\mathbb{R}$ bounded measurable, $\mathbb{E}[f(X)\mid\mathscr{E}]=\mathbb{E}[f(X)\mid\mathscr{T}]$.

3. Show that if $A\in\mathscr{E}$, then there is $B\in\mathscr{T}$ such that $A=B$ up to a $\mathbb{P}$-null set.

   Hint. Show that $\mathbb{P}(A\mid\mathscr{T})=\mathbb{1}_A$.

4. Suppose $X\in\{0,1\}^{\mathbb{N}}$. Compute $\mathbb{P}(X_1=x_1,\dots,X_k=x_k\mid\mathscr{E}_n)$ for all $n,k\geqslant 1$, $x\in\{0,1\}^k$. Deduce that given $P:=\mathbb{P}(X_1=1\mid\mathscr{E})$, the X_n, $n\geqslant 1$, are i.i.d. Bernoulli(P) r.v.

Exercise 2.7.16 (0-1 laws). Let $(\Omega,\mathscr{F},(\mathscr{F}_n),\mathbb{P})$ be a filtred probability space and $\mathscr{F}_\infty:=\bigvee_{n\geqslant 0}\mathscr{F}_n$.

1. a) Show that for every $X\in\mathrm{L}^1(\Omega,\mathscr{F},\mathbb{P})$,

$$\mathbb{E}[X\mid\mathscr{F}_n]\xrightarrow[n\to\infty]{}\mathbb{E}[X\mid\mathscr{F}_\infty],\quad\text{a.s. and in }\mathrm{L}^1.$$

   b) Deduce *Lévy's 0-1 law*: for every $A\in\mathscr{F}_\infty$,

$$\mathbb{P}(A\mid\mathscr{F}_n)\xrightarrow[n\to\infty]{}\mathbb{1}_A,\quad\text{a.s.}$$

2. Let $(X_n)_{n\geqslant 1}$ be a sequence of i.i.d. real r.v.

   a) Show *Kolmogorov's 0-1 law*: the tail σ-algebra $\mathscr{T}:=\bigcap_{n\geqslant 1}\sigma(X_k\colon k\geqslant n)$ is $\mathbb{P}$-trivial:

$$\forall A\in\mathscr{T},\quad\mathbb{P}(A)\in\{0,1\}.$$

   Hint. Use Lévy's 0-1 law.

   b) Use Kolmogorov's 0-1 law and Exercise 2.7.15.3 to reprove *Hewitt–Savage's 0-1 law*: the exchangeable σ-algebra $\mathscr{E}:=\sigma(f(X)\colon f\in\mathbf{S})$, where $\mathbf{S}:=\{f:\mathbb{R}^{\mathbb{N}}\to\mathbb{R}\text{ symmetric}\}$, is $\mathbb{P}$-trivial.

2.8 Markov chains

Exercise 2.8.1. Let $p\in(0,1)$, $X_1,X_2,\dots$ i.i.d. Bernoulli(p) r.v., and $S_n:=X_1+\cdots+X_n$. Justify whether each of the following processes is a Markov chain or not; if it is, give the corresponding state space E and the transition matrix Q.

1. X_n, $n\geqslant 0$;

2. S_n, $n\geqslant 0$;

3. $T_n:=S_1+\cdots+S_n$, $n\geqslant 0$;

4. $\mathbf{V}_n := (S_n, T_n)$, $n \geqslant 0$.

Exercise 2.8.2. Let $p \in (0,1)$ and $(X_n)_{n\geqslant 0}$ be a Markov chain on $E := \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ with transition matrix

$$Q := \begin{pmatrix} 1-p & p & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 0 & 1 \end{pmatrix}.$$

1. Draw its transition graph.
2. Compute the probability $\mathbb{P}(X_n = \mathbf{b} \mid X_0 = \mathbf{a})$, $n \in \mathbb{N}$. Find its limit as $n \to \infty$.

Exercise 2.8.3. Let $f : E \to F$ be a function between countable sets, and let $(X_n)_{n\geqslant 0}$ be a Markov chain on E with transition matrix P.

1. Find a simple counterexample showing that $Y_n := f(X_n)$, $n \geqslant 0$, is not necessarily a Markov chain on F.
2. We suppose that whenever $f(x) = f(y)$, then $P(x, A) = P(y, A)$ for every $A \subseteq E$. Show that $(Y_n)_{n\geqslant 0}$ is a Markov chain; express its transition matrix using P and f.

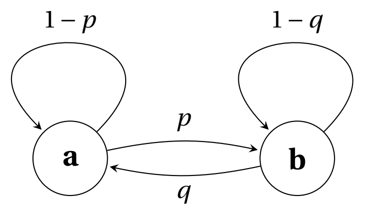
Exercise 2.8.4. Let $(U_n)_{n\geqslant 1}$ be i.i.d. uniform r.v. on $(0,1)$ and X_0 an independent r.v. on E.

1. Let $f : E \times (0,1) \to E$ and define $X_{n+1} := f(X_n, U_{n+1})$, $n \geqslant 0$. Show that $(X_n)_{n\geqslant 0}$ is a Markov chain on E. Express its transition matrix in terms of f and U_1.
2. Conversely, let P be a given transition matrix. Find a function $f : E \times (0,1) \to E$ such that the Markov chain $(X_n)_{n\geqslant 0}$ above has transition matrix P.

Exercise 2.8.5. Let E be a *finite* set of cardinal $k \geqslant 2$, and P be a transition matrix on E such that $\alpha := \inf\{P(x,y) : x, y \in E\} > 0$ (note then that $0 < \alpha \leqslant 1/2$).

1. We fix $y \in E$ and set $p_n(x) := P_n(x,y)$, $x \in E$.
   a) Show that for every $n \geqslant 0$,
   $$\begin{cases} \inf p_{n+k} \geqslant \alpha \sup p_n + (1-\alpha) \inf p_n, \\ \sup p_{n+k} \leqslant \alpha \inf p_n + (1-\alpha) \sup p_n. \end{cases}$$
   Hint. Use that $\sum_{x\in X} P_k(\cdot, x) + \sum_{x \in E\setminus X} P_k(\cdot, x) = 1$ for $X := \{x \in E : p_n(x) = \sup p_n\}$.
   b) Deduce that $d_n := \sup p_n - \inf p_n$ converges to 0 as $n \to \infty$.
2. Conclude that there exists a probability distribution $(p(y))_{y\in E}$ on E such that
   $$\forall x \in E, \quad p(y) = \lim_{n\to\infty} P_n(x,y).$$

Exercise 2.8.6. Let $p, q \in [0,1]$ and $(X_n)_{n\geqslant 0}$ be a Markov chain on $E := \{\mathbf{a}, \mathbf{b}\}$ with graph



1. For which values of p, q is $(X_n)_{n\geqslant 0}$ irreducible? Give the state classification.

2. Give the transition matrix of $(X_n)_{n\geqslant 0}$ and find the invariant probability measures.

3. Determine explicitly the law of X_n under $\mathbb{P}_{\mathbf{a}}$, for all $n \geqslant 0$.

4. Does $(X_n)_{n\geqslant 0}$ converge in law?

Exercise 2.8.7. Let $p \in [0,1]$, $q := 1-p$, and $(Y_k)_{k\geqslant 1}$ be a sequence of i.i.d. r.v. with $\mathbb{P}(Y_1 = 1) = p$, $\mathbb{P}(Y_1 = -1) = q$. Define $(X_n)_{n\geqslant 0}$ by $X_0 \in \mathbb{Z}_+$ and

$$X_{n+1} := (X_n + Y_{n+1})^+, \qquad n \geqslant 0,$$

where $x^+ := \max(x, 0)$.

1. Prove that $(X_n)_{n\geqslant 0}$ is a Markov chain. Give its state space and transition graph.

2. Is $(X_n)_{n\geqslant 0}$ irreducible? Give the state classification. (Discuss according to p.)

   Hint. Compare $(X_n)_{n\geqslant 0}$ to the random walk $\widetilde{X}_n := Y_1 + \cdots + Y_n$, $n \geqslant 0$, on $\mathbb{Z}$.

3. Determine all invariant measures of $(X_n)_{n\geqslant 0}$. Is there some invariant law?

Exercise 2.8.8. Let $(X_n)_{n\geqslant 0}$ be a Markov chain on a finite or countable state space E, and μ be a probability distribution on E.

1. Show that if X_n converges in law to μ as $n \to \infty$, then μ is an invariant measure.

2. Show that if μ is an invariant measure, then $\mu(x) = 0$ for all transient state $x \in E$.

Exercise 2.8.9. Suppose that we shuffle a traditional deck of 52 cards in the following way: at each time $n \in \mathbb{N}$, we choose two cards uniformly at random and exchange them.

1. Model this process by a Markov chain. (Give its state space and transition matrix.)

2. Show that this chain is irreducible and find its unique invariant distribution.

Exercise 2.8.10. Let $(X_n)_{n\geqslant 0}$ be a Markov chain on a finite or countable state space E. Recall that $H_x := \inf\{n \geqslant 1 \colon X_n = x\}$, $x \in E$, and, when x is recurrent, that

$$\nu_x(y) := \mathbb{E}_x\left[\sum_{n=0}^{H_x - 1} \mathbb{1}_{\{X_n = y\}}\right], \qquad y \in E$$

(the mean number of visits of y before returning to x), defines an invariant measure.

1. We suppose in this question that $(X_n)_{n\geqslant 0}$ is the symmetric random walk on $E=\mathbb{Z}$. Show that $\nu_0\equiv 1$ (the mean number of visits of $y\in\mathbb{Z}$ before returning to 0 is 1).

2. We suppose in this question that $(X_n)_{n\geqslant 0}$ is irreducible and positive recurrent. Show that for every $x,y\in E$,
$$\nu_x(y)=\frac{\mathbb{E}_x[H_x]}{\mathbb{E}_y[H_y]}.$$

Exercise 2.8.11. Let $(X_n)_{n\geqslant 0}$ be a Markov chain on a finite or countable state space E. Recall that $H_x:=\inf\{n\geqslant 1\colon X_n=x\}$, $x\in E$, and, when x is recurrent, that
$$\nu_x(y):=\mathbb{E}_x\left[\sum_{n=0}^{H_x-1}\mathbb{1}_{\{X_n=y\}}\right],\qquad y\in E$$
(the mean number of visits of y before returning to x), defines an invariant measure.

1. We suppose in this question that $(X_n)_{n\geqslant 0}$ is the symmetric random walk on $E=\mathbb{Z}$. Show that $\nu_0\equiv 1$ (the mean number of visits of $y\in\mathbb{Z}$ before returning to 0 is 1).

2. We suppose in this question that $(X_n)_{n\geqslant 0}$ is irreducible and positive recurrent. Show that for every $x,y\in E$,
$$\nu_x(y)=\frac{\mathbb{E}_x[H_x]}{\mathbb{E}_y[H_y]}.$$

Exercise 2.8.12. Let $(X_n)_{n\geqslant 0}$ be a Markov chain on $E:=\mathbb{Z}$ with transition matrix
$$Q(i,j):=\begin{cases} p_i, & \text{if } j^+=i^++1 \text{ or } j^-=i^-+1,\\ q_i, & \text{if } j^+=i^+-1 \text{ or } j^-=i^--1,\\ 0, & \text{otherwise,}\end{cases}$$
where $p_i\in(0,1)$, $q_i:=1-p_i$, for every $i\in E$.

1. Check that $(X_n)_{n\geqslant 0}$ is irreducible. (Sketch the transition graph.)

2. We suppose that
$$\limsup_{|k|\to\infty} p_k<\frac{1}{2}.$$
Show that $(X_n)_{n\geqslant 0}$ is (positive) recurrent.

   Hint. Apply Foster–Lyapunov's criterion.

Exercise 2.8.13. Let $(X_n)_{n\geqslant 0}$ be an irreducible Markov chain on E. We suppose that there exist a finite subset $F\subseteq E$ and a function $f\colon E\to\mathbb{R}$ such that

$$\text{(i)}\ \ \forall x\in F,\ f(x)>0;\quad \text{(ii)}\ \ \inf_{x\in E}f(x)=0;\quad \text{(iii)}\ \ \forall x\in E\setminus F,\ \mathbb{E}_x[f(X_1)]\leqslant f(x).$$

Show that $(X_n)_{n\geqslant 0}$ is transient.

Hint. Introduce the hitting time $T_F:=\inf\{n\geqslant 0\colon X_n\in F\}$...

Exercise 2.8.14. Let Q be a *symmetric*, irreducible, aperiodic transition matrix on E, and μ be a probability measure on E such that $\mu(x) > 0$ for all $x \in E$. We set

$$P(x,y) := Q(x,y)\min\left(1, \frac{\mu(y)}{\mu(x)}\right), \qquad \text{for } x \neq y \in E.$$

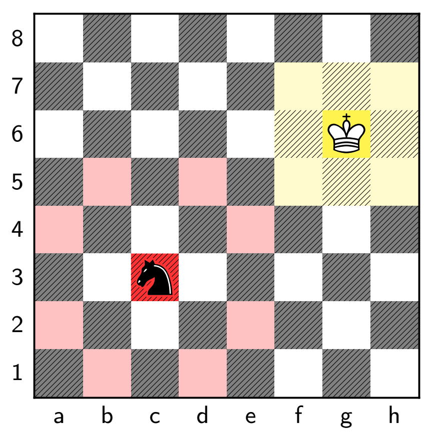
1. Check that P extends to a transition matrix which is also irreducible and aperiodic.

We consider a Markov chain $(X_n)_{n\geqslant 0}$ on E with transition matrix P.

2. Show that μ is an invariant measure for P, and that $(X_n)_{n\geqslant 0}$ is positive recurrent.

3. Let U_n, $n \in \mathbb{N}$, be a r.v. independent of $(X_k)_{k\geqslant 0}$, with $\mathbb{P}(U_n = i) = 1/n$, $0 \leqslant i < n$. Show that for every $x \in E$,
$$\sum_{x\in E} \left|\mathbb{P}(X_{U_n} = x) - \mu(x)\right| \xrightarrow[n\to\infty]{} 0.$$

Exercise 2.8.15. We consider the simple random walks of the knight and the king on a classical chessboard, $E := \{\mathsf{a}, \ldots, \mathsf{h}\} \times \{1, \ldots, 8\}$. Authorized moves are recalled below.



1. Starting in a8, what is the expected time for the king to return to a8? In the meantime, how many visits in the four squares {d4, e4, e5, d5} will he have performed?

   Hint. Use Exercise 2.8.11.2.

2. At which frequency does the knight visit square g6, as time tends to infinity?

Exercise 2.8.16. Let $(X_n)_{n\geqslant 0}$ be a Markov chain on a *finite* state space E, with transition matrix Q. We call a state $x \in E$ *absorbing*, and we write $x \in A$, if $Q(x,x) = 1$. We suppose $r := \sharp A \geqslant 1$ and A *accessible*: $\forall x \in E$, $\exists n \in \mathbb{N}$, $Q^n(x, A) > 0$.

1. Is $(X_n)_{n\geqslant 0}$ irreducible?

2. Let I_r denote the $r \times r$ identity matrix. Check that we may write Q in the form
$$Q := \left(\begin{array}{c|c} P & T \\ \hline 0 & I_r \end{array}\right).$$

3. Let $H_A := \inf\{n > 0 \colon X_n \in A\}$.

   a) Show that for all $i, j \notin A$, $P^n(i,j) \leqslant \mathbb{P}_i(H_A > n)$.

   b) Show that there exists $M \geqslant 1$ such that

$$p := \sup_{i \notin A} \mathbb{P}_i(H_A > M) < 1.$$

   Hint. You can take $M := \sup_{i \notin A} m_i$, where $m_i := \inf\{n > 0 \colon \mathbb{P}_i(X_n \in A) > 0\}$.

   c) Deduce that $\mathbb{P}_i(H_A = \infty) = 0$ and $P^n(i,j) \to 0$ for all $i, j \notin A$.
   Hint. Check that $\sup_{i \notin A} \mathbb{P}_i(H_A \geqslant Mn) \leqslant p^n$ (use the Markov property).

4. Let $s := \sharp E - r$. Show that $I_s - P$ is invertible and that, for $F := (I_s - P)^{-1}$,

$$\lim_{n\to\infty} Q^n = \left(\begin{array}{c|c} 0 & FT \\ \hline 0 & I_r \end{array}\right).$$

   Hint. Prove that 1 is not an eigenvalue of P.

5. a) Check that for all $i, j \notin A$,

$$F(i,j) = \mathbb{E}_i\left[\sum_{n=0}^{\infty} \mathbb{1}_{\{X_n = j\}}\right].$$

   b) Show that $\sum_{j \notin A} F(i,j) = \mathbb{E}_i[H_A]$ for all $i \notin A$.

   c) Show that $FT(i,j) = \mathbb{P}_i(X_{H_A} = j)$ for all $i \notin A$ and $j \in A$.

Exercise 2.8.17. Consider the Markov chain on $E := \{1,2,3,4,5,6\}$ with transition matrix:

$$Q := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0.4 & 0.6 & 0 & 0 & 0 & 0 \\ 0.3 & 0 & 0.4 & 0.2 & 0.1 & 0 \\ 0 & 0 & 0 & 0.3 & 0.7 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0.8 & 0 & 0.2 \end{pmatrix}.$$

1. Draw the transition graph.

2. Give the recurrence/transience classes.

3. Compute $\mathbb{P}_3(X_n \in \{4,5,6\}$ eventually$)$.
   Hint. Use the Markov property.

Exercise 2.8.18. Let $(Y_n)_{n\geqslant 0}$ be the symmetric random walk on $\mathbb{Z}$, that is $Y_n = Y_0 + \sum_{i=1}^n \xi_i$, $n \geqslant 0$, with ξ, $i \geqslant 1$, i.i.d. uniform ± 1 r.v. independent of $Y_0 \in \mathrm{L}^1$. Let $H_{-1} := \inf\{n > 0 \colon Y_n = -1\}$.

1. Let $k \in \mathbb{Z}_+$.

   a) What is $\mathbb{P}_0(H_{-1} = 2k)$?

b) Compute $\mathbb{P}_0(Y_{2k+1} = -1)$.

Hint. Under $\mathbb{P}_0$, $\{Y_{2k+1} = -1\}$ means that exactly k of the $\xi_1, \dots, \xi_{2k+1}$ equal $+1$...

c) Let $(x_i) \in \{\pm 1\}^{2k+1}$ with $x_1 + \cdots + x_{2k+1} = -1$. Check that there is one and only one $1 \leqslant r \leqslant 2k+1$ such that, if we set $\widetilde{x} := (x_{r+1}, \dots, x_{2k+1}, x_1, \dots, x_r)$, then

$$\forall j \leqslant 2k, \quad \sum_{i=1}^{j} \widetilde{x}_i \geqslant 0.$$

Suggestion. Do a drawing.

d) Deduce that $\mathbb{P}_0(H_{-1} = 2k+1) = \frac{1}{2k+1} \mathbb{P}_0(Y_{2k+1} = -1)$.

2. Give an equivalent of $\mathbb{P}_0(H_{-1} = 2k+1)$ as $k \to \infty$.

   Hint. Use Stirling's formula.

3. Conclude that $\mathbb{E}_0[H_{-1}] = \infty$.

Exercise 2.8.19. Using Exercise 2.7.11.5, give a simple proof that every irreducible, centered, finite-range random walk on $\mathbb{Z}$ is recurrent.

CHAPTER 3

COMBINATORICS OF INTEGER PARTITIONS

The following exercises are due to Jehanne Dousse.

3.1 Generating functionology

Exercise 3.1.1. List all partitions of 6.

Exercise 3.1.2.

1. List all partitions of 6 into even parts, and those in which each part occurs an even number of times. What do you notice?

2. Explain why, for n odd,

$$p(n \mid \text{even parts}) = p(n \mid \text{each part occurs an even number of times}) = 0.$$

3. Show that for all $n \in \mathbb{N}$,

$$p(n \mid \text{even parts}) = p(n \mid \text{each part occurs an even number of times}).$$

Exercise 3.1.3.

1. What is the generating function for partitions into distinct parts equal to 2, 5 or 7?

2. What is the generating function for partitions into parts equal to 2, 5 or 7, such that each part occurs at most d times ($d \in \mathbb{N}$)?

3. What is the generating function for partitions into parts equal to 2, 5 or 7?

Exercise 3.1.4. What generating function would you compute and what coefficient would you extract if you wanted to know the number of ways of changing a 100 CHF bill into coins of 1, 2 and 5 CHF and bills of 10 and 20 CHF?

Exercise 3.1.5. What is the generating function for partitions into parts $\leqslant 2k$ ($k \in \mathbb{N}$) where odd parts cannot repeat?

Exercise 3.1.6. Give the generating function for

$$\left(\frac{n^2+4n+5}{n!}\right)_{n\geqslant 0}.$$

Exercise 3.1.7.

1. Show that if f is the generating function for $(a_n)_{n\geqslant 0}$, then $\frac{f}{1-X}$ is the generating function for $(\sum_{j=0}^{n} a_j)_{n\geqslant 0}$.
2. Give the generating function for
$$\left(\sum_{j=0}^{n} j\right)_{n\geqslant 0}.$$
3. Show that if f is the generating function for $(a_n)_{n\geqslant 0}$, then f^k is the generating function for
$$\left(\sum_{n_1+\cdots+n_k=n} a_{n_1}\cdots a_{n_k}\right)_{n\geqslant 0}.$$
4. Recover the classical formula
$$\sum_{j=0}^{n} j = \frac{n(n+1)}{2}.$$

Exercise 3.1.8. Prove that for all $n \geqslant 0$,

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Hint. Compute a well-chosen generating function.

Exercise 3.1.9. Let $(a_n)_{n\geqslant 0}$ be a sequence defined by $a_0 = 0$ and for all $n \geqslant 1$,

$$a_n = 2a_{n-1} + 1.$$

What is the generating function for $(a_n)_{n\geqslant 0}$?

Exercise 3.1.10. Let $(b_n)_{n\geqslant 0}$ be a sequence defined by $b_0 = 1$ and for all $n \geqslant 1$,

$$b_n = 2b_{n-1} + n - 1.$$

1. What is the generating function for $(b_n)_{n\geqslant 0}$?

2. Give an explicit formula for b_n.

Exercise 3.1.11. We saw that if f is the generating function for $(a_n)_{n\geqslant 0}$, then $f/(1-X)$ is the generating function for $(\sum_{j=0}^{n} a_j)_{n\geqslant 0}$.
Use this to prove that the Fibonacci numbers f_n satisfy, for all $n \geqslant 0$,

$$f_0 + f_1 + \cdots + f_n = f_{n+2} - 1$$

Exercise 3.1.12. Let $c_{n,k}$ denote the number of compositions of n into k (nonzero) parts.

1. What is the (univariate) generating function for $(c_{n,k})_{n\geqslant 0}$?

2. Give an exact formula for $c_{n,k}$. You may use the formula

$$\sum_{n\geqslant 0} \binom{n}{k} X^n = \frac{X^k}{(1-X)^{k+1}}.$$

   Can you give a combinatorial interpretation?

3. What is the (bivariate) generating function for $(c_{n,k})_{k\geqslant 0}$?

4. Deduce an exact formula for c_n, the number of compositions of n. Can you give a combinatorial interpretation?

Exercise 3.1.13.

1. Show that

$$\sum_{n,k\geqslant 0} p(n \mid k \text{ parts, parts} \equiv j \bmod m)\, q^n z^k = \frac{1}{(zq^j; q^m)_\infty},$$

   and

$$\sum_{n,k\geqslant 0} Q(n \mid k \text{ parts, parts} \equiv j \bmod m)\, q^n z^k = (-zq^j; q^m)_\infty.$$

2. For n, k, m nonnegative integers, let $a(n, k, m)$ denote the number of partions of n into k distinct parts congruent to 2 mod 3 and m parts congruent to 1 mod 6, such that 2 is not a part. What is the (triviariate) generating function for $a_{n,k,m}$?

Exercise 3.1.14. Using generating functions, show that the number of partitions of n into parts congruent to ± 1 mod 6 equals the number of partitions of n into distinct parts congruent to ± 1 mod 3.

Exercise 3.1.15 (a bit challenging). Prove that the number of partitions of n such that each part appears 2, 3 or 5 times equals the number of partitions of n into parts congruent to ± 2, ± 3, or 6 mod 12.

Exercise 3.1.16. Show that for all $n, k \geqslant 1$,

$$p(n \mid k \text{ parts}) = p(n-1 \mid k-1 \text{ parts}) + p(n-k \mid k \text{ parts}).$$

3.2 Ferrers diagrams and q-series identities

Exercise 3.2.1. Find the conjugates of the following partitions:

- $6+6+4+2$,
- $3+3+2+1$,
- $6+1$.

Exercise 3.2.2. Use conjugation to show that for all n,

$$p(n \mid \text{distinct parts}) = p(n \mid \text{parts of every size from 1 to the largest part}).$$

For example, for $n=5$, the partitions of the first type are 5, $4+1$ and $3+2$, and the partitions of the second type are $1+1+1+1+1$, $2+1+1+1$ and $2+2+1$.

Exercise 3.2.3. Show that for all n, the number of partitions of n which have nothing under the Durfee square equals the number of partitions of n such that consecutive parts differ by at least 2.

Exercise 3.2.4. Using Ferrers diagrams, show that

$$\frac{1}{(zq;q)_\infty} = \sum_{n\geqslant 0}\left(\frac{z^n\,q^{2n^2}}{(q;q)_n\,(zq;q)_{2n}} + \frac{z^{n+1}\,q^{(n+1)(2n+1)}}{(q;q)_n\,(zq;q)_{2n+1}}\right).$$

Exercise 3.2.5. Show that for every $n\in\mathbb{N}$,

$$\sum_{k\in\mathbb{Z}}(-1)^k\,Q\left(n-\frac{k(3k+1)}{2}\right) = \begin{cases}(-1)^j, & \text{if } n=j(3j+1),\ j\in\mathbb{Z},\\ 0, & \text{otherwise.}\end{cases}$$

Hint. Use the pentagonal numbers theorem.

Exercise 3.2.6. Show that

$$\sum_{n\geqslant 0} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}.$$

Hint. Use Jacobi's triple product identity.

Exercise 3.2.7. We will now use Euler's Pentagonal numbers theorem to find the recurrence relation for $p(n)$ that was mentioned in class.

1. Show that

$$\left(\sum_{k\in\mathbb{Z}}(-1)^k\,q^{k(3k+1)/2}\right)\cdot\left(\sum_{n\geqslant 0}p(n)\,q^n\right) = 1.$$

2. Deduce that for every $n\in\mathbb{N}$,

$$p(n) = \sum_{k\geqslant 1}(-1)^{k-1}\,p\left(n-\frac{k(3k\pm 1)}{2}\right).$$

3. Write $p(10)$ as a sum of smaller values of $p(n)$.

(This method, discovered by Leonhard Euler in the 18th century, is still the fastest way to compute $p(n)$ and is used in computing softwares such as Maple, Mathematica, etc.)

Exercise 3.2.8. Prove the second q-analogue of Pascal's triangle.

Exercise 3.2.9. Give an analytic proof of the q-binomial series

$$\frac{1}{(zq;q)_n} = \sum_{m\geqslant 0} z^m q^m \begin{bmatrix} n+m-1 \\ m \end{bmatrix}_q.$$

Exercise 3.2.10. Show that for all integers $m, n \geqslant 0$,

$$\sum_{j=0}^{n} q^j \begin{bmatrix} m+j \\ m \end{bmatrix}_q = \begin{bmatrix} n+m+1 \\ m+1 \end{bmatrix}_q.$$

Exercise 3.2.11. Let

$$H_n(t) := \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q t^j.$$

Prove that

$$\sum_{n\geqslant 0} \frac{H_n(t)\, x^n}{(q;q)_n} = \frac{1}{(x;q)_\infty (xt;q)_\infty}.$$

Exercise 3.2.12. Show that, if n is odd,

$$\sum_{j=0}^{n} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix}_q = 0.$$

What happens if n is even?

Exercise 3.2.13. Let n tend to infinity in the q-binomial series

$$\frac{1}{(zq;q)_n} = \sum_{m\geqslant 0} z^m q^m \begin{bmatrix} n+m-1 \\ m \end{bmatrix}_q.$$

1. What do we obtain?
2. Give a combinatorial interpretation of the obtained formula.

Exercise 3.2.14. Show that

$$\sum_{n\geqslant 0} (-1)^n (2n+1)\, q^{\frac{n(n+1)}{2}} = (q;q)_\infty^3.$$

3.3 Congruence identities

Exercise 3.3.1. Prove the second Ramanujan congruence: for every $n \geqslant 0$,

$$p(7n+5) \equiv 0 \mod 7.$$

Exercise 3.3.2. For $k \geqslant 2$, the number of partitions of n into parts not divisible by k equals the number of partitions of n where each part occurs at most $k-1$ times. Prove this:

1. analytically,

2. bijectively.

Exercise 3.3.3 (Lemma for Schur's theorem)**.** Let $\pi_m(n)$ count the number of partitions $\lambda := \lambda_1 + \cdots + \lambda_s$ of n such that

$$\lambda_1 \leqslant m \quad \text{and, for all } 1 \leqslant i < s, \quad \lambda_i - \lambda_{i+1} \geqslant \begin{cases} 4, & \text{if } \lambda_i \text{ divisible by } 3, \\ 3, & \text{otherwise.} \end{cases}$$

Then we have the relations

(i) $\pi_{3m+1}(n) = \pi_{3m}(n) + \pi_{3m-2}(n-3m-1)$,

(ii) $\pi_{3m+2}(n) = \pi_{3m+1}(n) + \pi_{3m-1}(n-3m-2)$,

(iii) $\pi_{3m+3}(n) = \pi_{3m+2}(n) + \pi_{3m-1}(n-3m-3)$.

Prove (ii) and (iii).

Exercise 3.3.4. Let $(a_n)_{n\geqslant 0}$ be a sequence such that $\lim\limits_{n\to\infty} a_n$ exists. Prove Abel's lemma:

$$\lim_{x\to 1^-} (1-x) \sum_{n\geqslant 0} a_n x^n = \lim_{n\to\infty} a_n.$$

Exercise 3.3.5 (Reverse bijection for Schur's theorem)**.**

1. Show that the transformation from P_1 to P_4 in Schur's theorem is equivalent to the following process: As long as there exists some number that is not at least 3 greater than the number below, subtract 3 from this number, add 3 to the number below, and exchange these two numbers. Example:

$$P_1 = \begin{matrix} 11 \\ 18 \\ 5 \\ 3 \end{matrix} \to \begin{matrix} 21 \\ 8 \\ 6 \\ 2 \end{matrix} \to \begin{matrix} 21 \\ 9 \\ 5 \\ 2 \end{matrix} = P_1'.$$

2. Show that the following process is the reverse bijection of the above: Start by splitting parts of P_4 that are multiple of 3 into pairs of parts differing by 1 or 2. Example:

$$P_4 = \begin{array}{l} 21 \\ 9 \\ 5 \\ 2 \end{array} \to \begin{array}{l} 11+10 \\ 5+4 \\ 5 \\ 2 \end{array} = P_4'.$$

We obtain a partition P_4' with no multiples of 3. Now as long as the smallest part of some pair is less than 3 greater than the part below, subtract 3 from the largest part of the pair, add 3 to the part below, and switch their positions. This process ends with a partition into parts that are not multiples of 3, where parts differing by at most two are paired up, starting from the smallest part. Example:

$$P_4' = \begin{array}{l} 11+10 \\ 5+4 \\ 5 \\ 2 \end{array} \to \begin{array}{l} 11+10 \\ 8 \\ 4+2 \\ 2 \end{array} \to \begin{array}{l} 11 \\ 10+8 \\ 5 \\ 2+1 \end{array} = \begin{array}{l} 11 \\ 18 \\ 5 \\ 3 \end{array} = P_1''.$$

Exercise 3.3.6 (Refinement of Schur's theorem, Gleissberg). The goal of this exercise is to prove the following refinement of Schur's theorem due to Gleissberg. Let $C(m,n)$ denote the number of partitions if n into m distinct parts congruent to 1 or 2 mod 3. Let $D(m,n)$ denote the number of partitions of n into m parts (*counting parts divisible by* 3 *twice*), where parts differ by at least 3 and no two consecutive multiples of 3 appear. Then for all $m,n \geqslant 0$, $C(m,n) = D(m,n)$.

1. Let $\pi_\ell(m,n)$ denote the number of partitions counted by $D(m,n)$ such that the largest part does not exceed ℓ. Prove that for all ℓ, m, n positive integers,

$$\begin{aligned}
\pi_{3\ell+1}(m,n) &= \pi_{3\ell}(m,n) + \pi_{3\ell-2}(m-1, n-3\ell-1), \\
\pi_{3\ell+2}(m,n) &= \pi_{3\ell+1}(m,n) + \pi_{3\ell-1}(m-1, n-3\ell-2), \\
\pi_{3\ell+3}(m,n) &= \pi_{3\ell+2}(m,n) + \pi_{3\ell-1}(m-2, n-3\ell-3).
\end{aligned}$$

2. Define, for $|q|<1$, $|t|<1$,

$$a_\ell(t,q) := \sum_{m,n\geqslant 0} \pi_\ell(m,n)\, t^m q^n.$$

   What is $\lim_{\ell\to\infty} a_\ell(t,q)$?

3. Prove that

$$a_{3\ell-1}(tq^3,q) = (1 + tq^{3\ell+1} + tq^{3\ell+2})\, a_{3\ell-4}(tq^3,q) + t^2 q^{3\ell+3}(1-q^{3\ell-3})\, a_{3\ell-7}(tq^3,q).$$

4. Show that

$$a_{3\ell+3}(t,q) = (1 + tq^{3\ell+1} + tq^{3\ell+2})\, a_{3\ell}(t,q) + t^2 q^{3\ell+3}(1-q^{3\ell-3})\, a_{3\ell-3}(t,q).$$

5. What are the initial values $a_{-1}(tq^3, q)$, $a_2(tq^3, q)$, $a_3(t, q)$, $a_6(t, q)$? Verify that

$$a_3(t, q) = (1+tq)(1+tq^2)\, a_{-1}(tq^3, q),$$

and

$$a_6(t, q) = (1+tq)(1+tq^2)\, a_2(tq^3, q).$$

6. Deduce that for all $\ell \geqslant 0$,

$$a_{3\ell+3}(t, q) = (1+tq)(1+tq^2)\, a_{3\ell-1}(tq^3, q).$$

7. Conclude by finding $\lim_{\ell\to\infty} a_\ell(t, q)$.

Exercise 3.3.7. Let $M(k, r, n)$ denote the number of partitions of n with crank congruent to k modulo r. Show that for all $n \geqslant 0$,

$$M(0, 7, 7n+5) = \cdots = M(6, 7, 7n+5).$$

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