

Maximal correlation and monotonicity of free entropy and of Stein discrepancy

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Abstract

We introduce the maximal correlation coefficient $R(M_1, M_2)$ between two noncommutative probability subspaces M_1 and M_2 and show that the maximal correlation coefficient between the sub-algebras generated by $s_n := x_1 + \dots + x_n$ and $s_m := x_1 + \dots + x_m$ equals $\sqrt{m/n}$ for $m \leq n$, where $(x_i)_{i \in \mathbb{N}}$ is a sequence of free and identically distributed noncommutative random variables. This is the free-probability analogue of a result by Dembo–Kagan–Shepp in classical probability. As an application, we use this estimate to provide another simple proof of the monotonicity of the free entropy and free Fisher information in the free central limit theorem. Moreover, we prove that the free Stein Discrepancy introduced by Fathi and Nelson is non-increasing along the free central limit theorem.

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1 Introduction and main result

Pearson’s correlation coefficient $\rho(X_1, X_2) := \text{cov}(X_1, X_2) / \sigma(X_1) \sigma(X_2)$ is a standard measure of (bivariate) dependency. The *maximal correlation* $R(X_1, X_2)$ between two random variables X_1 and X_2 is then naturally defined as the supremum of $\rho(f(X_1), g(X_2))$ over L^2 -functions f, g .

In 2001, Dembo, Kagan, and Shepp [7] investigated this maximal correlation for the partial sums $S_n := X_1 + \dots + X_n$ of i.i.d. random variables X_i . Namely, regardless of the X_i ’s distribution, they showed that

$$R(S_n, S_m) \leq \sqrt{\frac{m}{n}}, \quad m \leq n,$$

with equality if $\sigma(X_1) < \infty$.

The aim of this note is to prove the analogous statement in the context of free probability, a theory which was initiated by Voiculescu (see [17]) in the 1980’s and has since flourished into an established area with connections to several other fields. We

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refer to [10, 12] for an introduction on the subject and an extensive list of references. Before stating our main result, let us first recall the necessary framework.

Let (M, τ) be a noncommutative, faithful, tracial probability space, that is a unital $*$ -algebra M equipped with a $*$ -linear form $\tau: M \rightarrow \mathbb{C}$ where $\tau(1) = 1$ and for all $x, y \in M$, $\tau(xy) = \tau(yx)$ (trace property), $\tau(x^*x) \geq 0$ (non-negativity), and $\tau(x^*x) = 0$ if and only if $x = 0$ (faithfulness). Elements of M are called *noncommutative random variables*, and the law of $(x_1, \dots, x_n) \in M^n$ is the family $(\tau(x_{i_1} \cdots x_{i_r})) : r \geq 1, 1 \leq i_1, \dots, i_r \leq n$. Finally, we recall the notion of *freeness*, which is the counterpart of classical independence: we say that (unital) sub-algebras $M_1, M_2, \dots \subseteq M$ are *free* if “any alternating product of centered elements is centered”, i.e., $\tau(x_1 \cdots x_r) = 0$ whenever $\tau(x_j) = 0$ for all $1 \leq j \leq r$ and $x_j \in M_{i_j}$ with $i_1 \neq i_2, i_2 \neq i_3, \dots, i_{r-1} \neq i_r$. In this vein, noncommutative random variables are free if the respective sub- $*$ -algebras they span are free.

The inner product

$$\langle x, y \rangle := \tau(x^*y), \quad (x, y) \in M^2,$$

confers on M a natural L^2 -structure with norm $\|x\|_2 := \sqrt{\langle x, x \rangle}$, so we may w.l.o.g. consider instead its Hilbert space completion $L^2(M)$. More generally, $L^2(M')$ will denote the closure in $L^2(M)$ of the sub-algebra $M' \subseteq M$. If $M' = \mathbb{C}\langle x_1, \dots, x_n \rangle$ is the sub- $*$ -algebra of noncommutative polynomials in $x_1, \dots, x_n \in M$ and $x_1^*, \dots, x_n^* \in M$, we refer to $L^2(M')$ as $L^2(x_1, \dots, x_n)$.

Let $\text{cov}(x, y) := \langle x - \tau(x) \cdot 1, y - \tau(y) \cdot 1 \rangle$ and $\sigma(x)^2 := \text{cov}(x, x)$. Just like in classical probability, we can define the Pearson correlation coefficient between x and y by

$$\rho(x, y) := \frac{\text{cov}(x, y)}{\sigma(x)\sigma(y)}.$$

Note that we have $|\rho(x, y)| \leq 1$ by the Cauchy–Schwarz inequality. Given $M_1, M_2 \subseteq M$ two subspaces, we call

$$R(M_1, M_2) := \sup_{M_1 \times M_2} \rho$$

the *maximal correlation coefficient* between M_1 and M_2 . For $M_1 = L^2(x)$, $M_2 = L^2(y)$, we simply write $R(x, y)$, and we call it the maximum correlation between the noncommutative random variables x and y .

We are now ready to state our main result.

Theorem 1.1. *Let $(x_i)_{i \in \mathbb{N}}$ be a sequence of free, non-zero, identically distributed, noncommutative random variables, and let $s_n := x_1 + \cdots + x_n$. Then for any $m \leq n$, we have*

$$R(s_n, s_m) = \sqrt{\frac{m}{n}}.$$

As in the classical setting, the interesting feature of the above statement is its universality as it holds regardless of the distribution of the noncommutative random variables. A possible way to prove the above statement consists of using the microstate approach by approximating the law of each noncommutative random variable by that of random matrices. One then exploits the multidimensional version of the classical maximal correlation inequality to apply it for the corresponding random matrices (seen as vectors) before passing to the limit and deducing the above theorem. The drawback of this approach is that it won't allow the extension of Theorem 1.1 to the multidimensional case. Indeed, by the refutation of Connes embedding problem [11], there are noncommutative random variables whose joint moments cannot be approximated well by moments of matrices. This makes it impossible to use the mentioned approach to prove a multidimensional version of the above theorem. On the contrast, our proof of Theorem 1.1, which is carried in Section 2, adapts the approach of [7] to a noncommutative setting and is readily extendable to the multidimensional setting.

A celebrated result of Artstein et al [1] provided a solution to Shannon’s problem regarding the monotonicity of entropy in the classical central limit theorem. In the context of free probability, the concept of *free entropy* and *information* was developed by Voiculescu in a series of papers (see for example [20]). Two approaches were given for the definition of free entropy, referred to as *microstates* and *non-microstates* and denoted by χ and χ^* respectively (see [12, Chapters 7 and 8]). These two coincide in the one-dimensional setting, in which case the free entropy of a compactly supported probability measure μ is given by

$$\chi(\mu) = \chi^*(\mu) := \iint_{\mathbb{R}^2} \log |a - b| \mu(da)\mu(db) + \frac{3}{4} + \frac{1}{2} \log(2\pi).$$

It is not known whether χ and χ^* coincide in the multidimensional setting. Our proof of the result presented in the sequel extends to the multidimensional case for χ^* only. Given a noncommutative probability space (M, τ) and a self-adjoint element $z \in M$, we define $\chi^*(z)$ as $\chi^*(\mu_z)$ where μ_z denotes the distribution of z , i.e., the probability measure characterized by $\int p d\mu_z = \tau(p(z))$ for all polynomials $p \in \mathbb{C}[X]$.

In [15], Shlyakhtenko proved the monotonicity of the free entropy in the free central limit theorem providing an analogue of the result of [1] in the noncommutative setting. As an application of our maximal correlation estimate, we recover the monotonicity property which we state in the next corollary.

Corollary 1.2. *Given $(x_i)_{i \in \mathbb{N}}$ a sequence of free, identically distributed, self-adjoint random variables in (M, τ) , one has*

$$\chi^* \left(\frac{s_m}{\sqrt{m}} \right) \leq \chi^* \left(\frac{s_n}{\sqrt{n}} \right), \tag{1.1}$$

for every integers $m \leq n$.

As in the classical setting, the monotonicity of the entropy follows from that of the Fisher information. In Section 3, we prove the latter as a consequence of Theorem 1.1. The idea of using the maximal correlation inequality in this context goes back to Courtade [5] who used the result of Dembo, Kagan, and Shepp [7] to provide an alternative proof of the monotonicity of entropy in the classical setting.

Formulated alternatively, the above corollary states that given a compactly supported probability measure μ and positive integers $m \leq n$, one has

$$\chi^*(m^{-1/2} *_\mu^{\boxplus m}) \leq \chi^*(n^{-1/2} *_\mu^{\boxplus n}), \tag{1.2}$$

where \boxplus denotes the free convolution operation (so $\mu^{\boxplus n}$ is the distribution of the sum of n free copies of an element x with distribution μ), and α_* is the pushforward operation by the dilation $t \mapsto \alpha t$. As a matter of fact, it is possible to make sense of $\mu^{\boxplus t}$ for all real $t \geq 1$ (see [14]). Very recently, Shlyakhtenko and Tao [16] extended (1.2) to real-valued exponents while providing two different proofs. It would be interesting to see if the argument in this paper based on maximal correlation could be extended to cover non-integer exponents.

Another consequence of Theorem 1.1 concerns the monotonicity of the *free Stein discrepancy* along the free central limit theorem. Stein discrepancy measures in some sense how far is a probability measure from another one characterized by some integration by parts formula. Using the classical maximal correlation inequality, it was proven by Courtade, Fathi and Pananjady [6] that the Stein discrepancy (relative to the standard Gaussian measure) is non-increasing in the central limit theorem. The notion of free Stein discrepancy relative to a semicircular law was introduced by Fathi and Nelson [9]. Recall that a standard semicircular variable S is a self-adjoint element of (M, τ) whose

distribution has density $\frac{1}{2\pi}\sqrt{4-t^2}\mathbf{1}_{[-2,2]}(t)$. Analogously to the normal distribution, a standard semicircular variable $S \in M$ is characterized by the following integration by parts formula stating that

$$\langle S, P(S) \rangle = \langle 1 \otimes 1, \partial P(S) \rangle,$$

for every polynomial P . Here, ∂ denotes the noncommutative derivative and the right hand side dot product refers to the dot product in the Hilbert space $L^2(M) \otimes L^2(M)$ (see [12]). Following [9], a *free Stein kernel* of $x \in M$ is an element $K \in L^2(M) \otimes L^2(M)$ such that

$$\langle x, P(x) \rangle = \langle K, \partial P(x) \rangle,$$

for every polynomial P . It was shown by Cébron, Fathi and Mai [4] that free Stein kernel always exist if $\tau(x) = 0$. The free Stein discrepancy of x relative to S is then defined as

$$\Sigma^*(x | S) = \inf_K \|K - 1 \otimes 1\|_{L^2(M) \otimes L^2(M)},$$

where the infimum is taken over all free Stein kernels K of x . We should note that [9] introduced the notion of free Stein kernel/discrepancy relative to a general potential while we will only be dealing with the particular case of the potential $t^2/2$ leading to the notions stated above.

As an application of our maximal correlation inequality, we obtain the following corollary extending the aforementioned monotonicity of Stein discrepancy obtained in [6] to the free setting.

Corollary 1.3. *Given $(x_i)_{i \in \mathbb{N}}$ a sequence of free, centered, identically distributed, self-adjoint random variables in (M, τ) with unit norm, one has*

$$\Sigma^*\left(\frac{s_n}{\sqrt{n}} \mid S\right) \leq \sqrt{\frac{m}{n}} \Sigma^*\left(\frac{s_m}{\sqrt{m}} \mid S\right), \tag{1.3}$$

for every integers $m \leq n$.

Note that taking $m = 1$ in the above corollary, we obtain that $\Sigma^*(n^{-\frac{1}{2}}s_n | S)$ decays faster than C/\sqrt{n} for some constant C recovering a result of [4]. Similarly to the previous results of this paper, the proof of the above corollary works verbatim in the multidimensional setting.

2 Proof of Theorem 1.1

Let us fix $x_1, x_2, \dots \in M$. For every finite set $I \subset \mathbb{N}$, we denote

$$L_I^2 := L^2(x_i : i \in I) \quad \text{and} \quad \text{proj}_I(z) := \text{proj}_{L_I^2}(z)$$

the orthogonal projection of $z \in M$ onto L_I^2 , which is nothing else but the *conditional expectation* of z given L_I^2 :

$$y = \text{proj}_I(z) \iff y \in L_I^2 \quad \text{and} \quad \forall x \in L_I^2, \tau(xy) = \tau(xz).$$

In particular $\text{proj}_I(z) = z$ if $z \in L_I^2$, and by the trace property

$$\text{proj}_I(xzy) = x \text{proj}_I(z)y \quad \text{for all } x, y \in M \cap L_I^2, z \in M. \tag{2.1}$$

Note that $\text{proj}_\emptyset(z) = \tau(z) \cdot 1$ and $\text{proj}_J \circ \text{proj}_I = \text{proj}_J$ for every $J \subseteq I$ (tower property). When freeness is further involved, we can say a bit more. The following lemma appears in different forms in the literature ([2], [12, Section 2.5]), we include its proof for completeness.

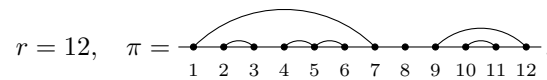
Lemma 2.1. *Let $I, J \subset \mathbb{N}$ be finite sets and suppose $(x_k : k \in I \cup J)$ is free. Then:*

- (i) *if z is a (noncommutative) polynomial in variables in $\mathbb{C}\langle x_i : i \in I \rangle$ and in variables in $\mathbb{C}\langle x_j : j \in J \setminus I \rangle$, then $\text{proj}_I(z)$ is a polynomial in only those variables in $\mathbb{C}\langle x_i : i \in I \rangle$;*
- (ii) *the projections commute: $\text{proj}_I \circ \text{proj}_J = \text{proj}_{I \cap J}$.*

Proof. (i) By linearity of proj_I , we may suppose without loss of generality that $z = a_1 \cdots a_r$ with each a_k in $\mathbb{C}\langle x_i : i \in I \rangle$ or $\mathbb{C}\langle x_j : j \in J \setminus I \rangle$. From the moment-cumulant formula [12, Definition 9.2.7] w.r.t. the conditional expectation proj_I , we can write

$$\text{proj}_I(z) := \sum_{\pi \in \text{NC}(r)} \kappa_{\pi}^I(a_1, \dots, a_r),$$

where the summation ranges over all non-crossing partitions $\pi \in \text{NC}(r)$ of $\{1, \dots, r\}$ and the κ_{π}^I are nestings (consistently with the blocks of π) of the free (conditional) cumulants $\kappa_n^I : M^n \rightarrow L_I^2$ (which can be defined inductively). For instance, if



then

$$\kappa_{\pi}^I(a_1, \dots, a_r) = \kappa_2^I(a_1 \cdot \kappa_2^I(a_2, a_3) \cdot \kappa_3^I(a_4, a_5, a_6), a_7) \cdot \kappa_1^I(a_8) \cdot \kappa_2^I(a_9 \cdot \kappa_2^I(a_{10}, a_{11}), a_{12}).$$

As the bimodularity property (2.1) extends to conditional cumulants [12, Remark 9.1], each of the innermost cumulants κ_s^I is in the form

$$b_0 \cdot \kappa_s^I(n_1 \cdot b_1, n_2 \cdot b_2, \dots, n_s) \cdot b_s,$$

where $s \in \mathbb{N}$ is some integer and the b_i 's (resp. n_j 's) are (possibly constant) polynomials in the subset of the variables a_1, \dots, a_r that belong to $\mathbb{C}\langle x_i : i \in I \rangle$ (resp. $\mathbb{C}\langle x_j : j \in J \setminus I \rangle$). We now invoke [13, Theorem 3.6] with $F := \tau|_{L_I^2}$ and the sub-algebra $N := \mathbb{C}\langle x_j : j \in J \setminus I \rangle$ free from $B := L_I^2$ over $D := \mathbb{C} \cdot 1$ to see that the κ_s^I term reduces to a constant:

$$\kappa_s^I(n_1 \cdot b_1, n_2 \cdot b_2, \dots, n_s) = \tau(b_1) \cdots \tau(b_{s-1}) \cdot \kappa_s(n_1, n_2, \dots, n_s) \in \mathbb{C},$$

where κ_s is the (unconditioned) cumulant $M^s \rightarrow \mathbb{C}$. By multilinearity of the cumulants, the outer nestings of κ_{π}^I remain in the same form, and reducing them successively thus shows that each term $\kappa_{\pi}^I(a_1, \dots, a_r)$ is in fact a polynomial only in the subset of the variables a_1, \dots, a_r that belong to $\mathbb{C}\langle x_i : i \in I \rangle$. This proves the claim.

- (ii) Clearly $\tau(x \text{proj}_I \circ \text{proj}_J(z)) = \tau(\text{proj}_I(\text{proj}_J(xz))) = \tau(xz)$ for all $x \in L_{I \cap J}^2$, so we only need to check that $\text{proj}_I(\text{proj}_J(z)) \in L_{I \cap J}^2$. By a closure argument, we may suppose that $\text{proj}_J(z)$ belongs to $\mathbb{C}\langle x_j : j \in J \rangle$. In this case $\text{proj}_I(\text{proj}_J(z)) \in \mathbb{C}\langle x_k : k \in I \cap J \rangle \subseteq L_{I \cap J}^2$ by the previous point. \square

We now set, for every $z \in M$,

$$z_I := \sum_{J \subseteq I} (-1)^{|I|-|J|} \text{proj}_J(z) \in L_I^2, \tag{2.2}$$

where $|\cdot|$ is the cardinal notation. The following Efron–Stein (a.k.a. ANOVA) decomposition will play a crucial role in the proof of Theorem 1.1. Its use in free probability seems new (to the best of the authors' knowledge) and may be of other independent interest.

Lemma 2.2. For every finite set $I \subset \mathbb{N}$,

$$\text{proj}_I(z) = \sum_{J \subseteq I} z_J.$$

Proof. We repeat in a compact way the argument of Efron and Stein [8]:

$$\begin{aligned} \sum_{J \subseteq I} z_J &= \sum_{J \subseteq I} \sum_{K \subseteq J} (-1)^{|J|-|K|} \text{proj}_K(z) \\ &= \sum_{K \subseteq I} \underbrace{\left(\sum_{K \subseteq J \subseteq I} (-1)^{|J|-|K|} \right)}_{=(1-1)^{|I \setminus K|}} \text{proj}_K(z) \\ &= \text{proj}_I(z). \quad \square \end{aligned}$$

The elements z_J will be orthogonal thanks to this direct consequence of Lemma 2.1:

Lemma 2.3. Suppose that x_1, x_2, \dots are free, and let $I, J \subset \mathbb{N}$ be finite sets such that $I \setminus J \neq \emptyset$. Then $\text{proj}_J(z_I) = 0$ for every $z \in M$. In particular, z_I is orthogonal to $z_J \in L_J^2$.

Proof. Apply Lemma 2.1 and gather the subsets $K \subseteq I$ that have same intersection $L := J \cap K$ with J :

$$\begin{aligned} \text{proj}_J(z_I) &= \sum_{K \subseteq I} (-1)^{|I|-|K|} \text{proj}_J \circ \text{proj}_K(z) \\ &= \sum_{L \subseteq I \cap J} (-1)^{|I|-|L|} \underbrace{\left(\sum_{K \subseteq I \setminus J} (-1)^{|K|} \right)}_{=(1-1)^{|I \setminus J|}} \text{proj}_L(z) \\ &= 0. \quad \square \end{aligned}$$

To prove Theorem 1.1 we shall finally exploit that the partial sum $s_n := x_1 + \dots + x_n$ is symmetric in (x_1, \dots, x_n) . Our next proposition is tailored for this purpose.

Proposition 2.4. Suppose that x_1, \dots, x_n are free and identically distributed. Then for every symmetric polynomial $z = p(x_1, \dots, x_n)$ in x_1, \dots, x_n and every $I \subseteq \{1, \dots, n\}$, the pair (z_I, z_I^*) has the same distribution as $(z_{I'}, z_{I'}^*)$ where $I' := \{1, \dots, |I|\}$.

Consequently, if $\tau(z) = 0$, then

$$\|\text{proj}_I(z)\|_2 \leq \sqrt{\frac{|I|}{n}} \|z\|_2.$$

Proof. Let $m = |I|$ and let ς be a permutation of $\{1, \dots, n\}$ mapping I' to I . Since p is symmetric, we can write $z = p(x_1, \dots, x_n) = p(x_{\varsigma(1)}, \dots, x_{\varsigma(n)})$. By Lemma 2.1, we see from its definition in (2.2) that z_I is a certain polynomial q , in the $x_i, i \in I$, that is

$$z_I = z_{\varsigma(I')} = q(x_{\varsigma(1)}, \dots, x_{\varsigma(m)}).$$

Now the x_i 's are free and identically distributed, so the relabelling shows that

$$z_{I'} = q(z_1, \dots, z_m)$$

with the same polynomial q , and thus the pairs

$$(z_I, z_I^*) = (q(x_{\varsigma(1)}, \dots, x_{\varsigma(m)}), q(x_{\varsigma(1)}, \dots, x_{\varsigma(m)})^*)$$

and

$$(z_{I'}, z_{I'}^*) = (q(x_1, \dots, x_m), q(x_1, \dots, x_m)^*)$$

have the same moments. In particular $\|z_{I'}\|_2 = \|z_{I'}^*\|_2$, and the stated inequality becomes clear by combining this with Lemmas 2.2 and 2.3:

$$\begin{aligned} \|\text{proj}_I(z)\|_2^2 &= \sum_{J \subseteq I} \|z_J\|_2^2 \\ &= \sum_{k=1}^m \binom{m}{k} \|z_{\{1, \dots, k\}}\|_2^2 \\ &\leq \frac{m}{n} \sum_{k=1}^n \binom{n}{k} \|z_{\{1, \dots, k\}}\|_2^2 \\ &= \frac{m}{n} \|\text{proj}_{\{1, \dots, n\}}(z)\|_2^2 \\ &= \frac{m}{n} \|z\|_2^2, \end{aligned}$$

using that $z_0 = 0$, that $\binom{m}{k} \leq \frac{m}{n} \binom{n}{k}$ for all $1 \leq k \leq m \leq n$, and that $z \in L^2(x_1, \dots, x_n)$. \square

Proof of Theorem 1.1. The lower bound $R(s_n, s_m) \geq \sqrt{m/n}$ is straightforward since, by freeness, $\sigma(s_n)^2 = n \sigma(x_1)^2$ and $\text{cov}(s_n, s_m) = \sigma(s_m)^2 = m \sigma(x_1)^2$. For the upper bound, we must show that $\rho(z, z') \leq \sqrt{m/n}$ for all $z \in L^2(s_n)$ and $z' \in L^2(s_m)$. W.l.o.g., we may suppose that $\tau(z) = \tau(z') = 0$ and, by another closure argument, that z is a polynomial in s_n (and thus a symmetric polynomial in x_1, \dots, x_n). Then by the Cauchy–Schwarz inequality and Proposition 2.4,

$$\begin{aligned} \text{cov}(z, z') &= \langle z, z' \rangle \\ &= \langle \text{proj}_{\{1, \dots, m\}}(z), z' \rangle \\ &\leq \|\text{proj}_{\{1, \dots, m\}}(z)\|_2 \|z'\|_2 \\ &\leq \sqrt{\frac{m}{n}} \|z\|_2 \|z'\|_2 \\ &= \sqrt{\frac{m}{n}} \sigma(z) \sigma(z'), \end{aligned}$$

and the proof is complete. \square

3 Monotonicity of the free entropy and free Fisher information

The goal of this section is to prove Corollary 1.2. Let us start by noting that the free entropy and free Fisher information of a self-adjoint element $z \in M$ are related through the integral formula ([18]; see also [12, Chapter 8])

$$\chi^*(z) = \frac{1}{2} \int_0^\infty \left(\frac{1}{1+t} - \Phi(z + \sqrt{t}S) \right) dt + \frac{1}{2} \log(2\pi e), \tag{3.1}$$

where S stands for a standard semicircular variable free from z , and Φ denotes the free Fisher information. After [18], the free Fisher information of a noncommutative, self-adjoint random variable $z \in M$ is defined as $\Phi(z) := \|\xi\|_2^2$ where the so called *conjugate variable* $\xi := \xi(z)$ is any element of $L^2(z)$ such that, for every integer $r \geq 0$,

$$\tau(\xi z^r) = \sum_{k=0}^{r-1} \tau(z^k) \tau(z^{r-1-k}). \tag{3.2}$$

(If such a ξ does not exist, we set $\Phi(z) := \infty$.) We note from (3.2) that $\tau(\xi(z)) = 0$ and the homogeneity property $\Phi(\alpha z) = \alpha^{-2} \Phi(z)$, $\alpha > 0$.

In the next Corollary, we show how the monotonicity of the free Fisher information follows easily from Theorem 1.1.

Corollary 3.1. *Let $(x_i)_{i \in \mathbb{N}}$ be a sequence of free, identically distributed, self-adjoint random variables in (M, τ) and denote $s_k := x_1 + \dots + x_k$ for every positive integer k . Then for all positive integers $m \leq n$, we have*

$$\Phi\left(\frac{s_n}{\sqrt{n}}\right) \leq \Phi\left(\frac{s_m}{\sqrt{m}}\right).$$

Proof. Assume the existence of $\xi(s_1)$, as otherwise $\Phi(s_1) = \infty$ and there is nothing to prove. According to [12, p. 206], the free sum $s_n = s_m + (s_n - s_m)$ admits $\xi(s_n) = \text{proj}_{L^2(s_n)}(\xi(s_m))$ as conjugate variable. Therefore, by Theorem 1.1,

$$\begin{aligned} \Phi(s_n) &= \|\text{proj}_{L^2(s_n)}(\xi(s_m))\|_2^2 = \text{cov}\left(\text{proj}_{L^2(s_n)}(\xi(s_m)), \xi(s_m)\right) \\ &\leq \sqrt{\frac{m}{n}} \|\text{proj}_{L^2(s_n)}(\xi(s_m))\|_2 \|\xi(s_m)\|_2, \end{aligned}$$

i.e., $\Phi(s_n) \leq \frac{m}{n} \Phi(s_m)$. We conclude by the homogeneity property. □

In view of (3.1) and the divisibility of the semicircular distribution w.r.t. the free convolution, the above corollary readily implies (1.1), thus proving Corollary 1.2.

4 Monotonicity of the free Stein discrepancy

The goal of this section is to provide a proof of Corollary 1.3. Let us fix $m \leq n$ and x_1, x_2, \dots a sequence of free, centered, identically distributed, self-adjoint random variables in (M, τ) with unit norm. Let us record the following consequence of Theorem 1.1 which will be used in the sequel.

Lemma 4.1. *Let $K \in L^2(s_m) \otimes L^2(s_m)$ and suppose that $\langle K, 1 \otimes 1 \rangle = 1$. Then*

$$\|\text{proj}_{L^2(s_n) \otimes L^2(s_n)}(K - 1 \otimes 1)\|_{L^2(s_n) \otimes L^2(s_n)} \leq \sqrt{\frac{m}{n}} \|K - 1 \otimes 1\|_{L^2(s_m) \otimes L^2(s_m)}.$$

Proof. Let us denote by Q the projection on 1 and write P_n (resp. P_m) for $\text{proj}_{L^2(s_n)}$ (resp. $\text{proj}_{L^2(s_m)}$). Note that using Theorem 1.1, we have for any $b \in L^2(s_m)$

$$\|(P_n - Q)(b)\|_2^2 = \langle (P_n - Q)(b), b - Q(b) \rangle \leq \sqrt{\frac{m}{n}} \|(P_n - Q)(b)\|_2 \|b - Q(b)\|_2,$$

which leads to

$$\|(P_n - Q)(b)\|_2 \leq \sqrt{\frac{m}{n}} \|b - Q(b)\|_2.$$

Using that $P_\ell Q = Q P_\ell = Q$ for $\ell \in \{n, m\}$, the above relation can be equivalently stated as

$$P_m P_n P_m - Q \preceq \frac{m}{n} (P_m - Q),$$

where \preceq denotes the positive semi-definite ordering. Using this we can write

$$\begin{aligned} P_m P_n P_m \otimes P_m P_n P_m - Q \otimes Q &= P_m P_n P_m \otimes (P_m P_n P_m - Q) + (P_m P_n P_m - Q) \otimes Q \\ &\preceq \frac{m}{n} P_m \otimes (P_m - Q) + \frac{m}{n} (P_m - Q) \otimes Q \\ &= \frac{m}{n} (P_m \otimes P_m - Q \otimes Q), \end{aligned}$$

where we have also used that $P_m P_n P_m \preceq P_m$. Note that since $\langle K, 1 \otimes 1 \rangle = 1$, then $K - 1 \otimes 1 = (P_m \otimes P_m - Q \otimes Q)(K)$. The proof is finished after observing that

$$(P_m \otimes P_m - Q \otimes Q)P_n \otimes P_n(P_m \otimes P_m - Q \otimes Q) = P_m P_n P_m \otimes P_m P_n P_m - Q \otimes Q,$$

and combining it with the above. □

Given $K \in L^2(s_m) \otimes L^2(s_m)$ a free Stein kernel of s_m/\sqrt{m} , we have by the definition of K (and homogeneity) that

$$\left\langle \frac{s_m}{\sqrt{m}}, P(s_m) \right\rangle = \sqrt{m} \langle K, \partial P(s_m) \rangle,$$

for every polynomial P . Since s_m and $s_n - s_m$ are free, then using the intertwining relation between ∂ and the conditional expectation [19], we can write

$$\left\langle \frac{s_m}{\sqrt{m}}, P(s_n) \right\rangle = \sqrt{m} \langle K, \partial P(s_n) \rangle.$$

The above relation appears in [3] (before Lemma 2.5 there). Using linearity and that the x_i 's are identically distributed, we get

$$\langle s_m, P(s_n) \rangle = m \langle x_1, P(s_n) \rangle = \frac{m}{n} \langle s_n, P(s_n) \rangle.$$

Putting the above relations together, we deduce that

$$\left\langle \frac{s_n}{\sqrt{n}}, P(s_n) \right\rangle = \sqrt{n} \langle K, \partial P(s_n) \rangle,$$

for every polynomial P . This implies that $\text{proj}_{L^2(s_n) \otimes L^2(s_n)}(K)$ is a free Stein kernel of s_n/\sqrt{n} . Thus, we have

$$\Sigma^* \left(\frac{s_n}{\sqrt{n}} \mid S \right) \leq \inf_K \left\| \text{proj}_{L^2(s_n) \otimes L^2(s_n)}(K - 1 \otimes 1) \right\|_{L^2(s_n) \otimes L^2(s_n)},$$

where the infimum is taken over all free Stein kernels $K \in L^2(s_m) \otimes L^2(s_m)$ of s_m/\sqrt{m} . Noting that $\langle K, 1 \otimes 1 \rangle = 1$, then using Lemma 4.1 we deduce

$$\Sigma^* \left(\frac{s_n}{\sqrt{n}} \mid S \right) \leq \sqrt{\frac{m}{n}} \inf_K \left\| K - 1 \otimes 1 \right\|_{L^2(s_m) \otimes L^2(s_m)},$$

where the infimum is again taken over all free Stein kernels $K \in L^2(s_m) \otimes L^2(s_m)$ of s_m/\sqrt{m} . Corollary 1.3 then follows after noting that the projection on $L^2(s_m) \otimes L^2(s_m)$ of every free Stein kernel of s_m/\sqrt{m} is also a free Stein kernel.

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