Monotonicity of the Logarithmic Energy for Random Matrices

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joint work with D. Chafaï (ENS Paris) and P. Youssef (NYUAD)

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Shannon's entropy in the central limit theorem

Shannon's classical entropy of a random variable X with density f:

$$\mathcal{S}(X) := -\mathbb{E}\log f(X) = -\int f \log f \quad (=: \mathcal{S}(f)).$$

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Shannon's problem (1940)

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$$Y_n := \frac{S_n}{\sqrt{n}} \xrightarrow[n\to\infty]{} G \sim \mathcal{N}(0,1).$$

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- Let $\xi_X := \frac{f'(X)}{f(X)}$ (score) and $\Phi(X) := \mathbb{E}\,\xi_X^2$ (Fisher information).

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- Dembo–Kagan–Shepp (2001): max-correlation $\sup_{\substack{f \in L^2(S_m)\\ g \in L^2(S_n)}} \rho(f,g) = \sqrt{\frac{m}{n}}$.
- Thus $n\,\Phi(S_n) \leq m\,\Phi(S_m)$, which means that $\Phi(Y_n) \leq \Phi(Y_m)$.

Other occurrences of "entropy" and monotonicity

- **Sanov's theorem**: if $(X_i)_{i\geq 1}$ are i.i.d. with law μ, then (LDP)

$$\mathbb{P}\left(\frac{1}{n}\sum_{k=1}^{n}\delta_{X_k}\in A\right)\approx \exp\left(-n\inf_A K\right),\ \text{ where } K(\nu):=\mathcal{S}\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right).$$

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- Boltzmann entropy in Boltzmann's equation of an isolated gas:

$$\partial_t f_t(x,v) = -v\partial_x f_t(x,v) + Q(f_t,f_t)(x,v),$$

with $f_t(x,v)$ the density of particles with speed v at position x.

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- Generally, for $\mathrm{d}X_t=\mathrm{d}B_t-\nabla V(X_t)\,\mathrm{d}t$ and $f_t\,\mathrm{d}x=\mathbb{P}(X_t\in\mathrm{d}x)$, the **Helmholtz free energy** decreases:

$$\mathcal{A}(f_t)\searrow \mathcal{A}(f_*)=\min\mathcal{A},\ \text{ with } \mathcal{A}(f):=-\mathcal{S}(f)+\int Vf.$$

For details: Villani, *H-theorem and beyond* (2006).

One last example: Voiculescu's free entropy

- Voiculescu's free entropy of a self-adjoint operator is

$$\Sigma(\mu) := \iint \log|x-y|\,\mu(\mathrm{d}x)\mu(\mathrm{d}y),$$

expressed in terms of its spectral measure μ.

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Voiculescu's free central limit theorem (1991)

If $x_1, x_2, \dots$ are free and identically distributed with $\tau(x_1) = 0$ and $\tau(x_1^2) = 1$, then

$$y_n := \frac{x_1 + \cdots + x_n}{\sqrt{n}} \xrightarrow[n\to\infty]{} s \text{ (semi-circular element)},$$

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- Shlyakhtenko–Tao (2021): $\Sigma(t^{-\frac{1}{2}}{}_*\mu^{\boxplus t})$ is increasing with $t > 0$.

What about the classical theorems of random matrices?

- For $A \in \mathbb{C}^{n\times n}$, we denote by μ_A its empirical spectral distribution:

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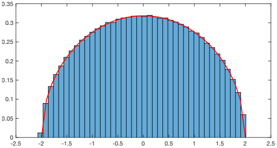

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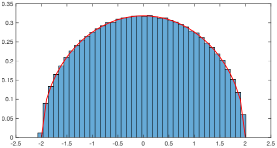

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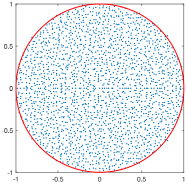

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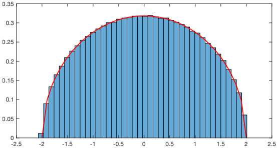

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$$W_n := \frac{X_n + X_n^*}{\sqrt{2n}}$$

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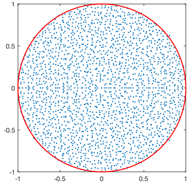

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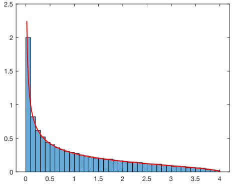

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What about the classical theorems of random matrices?

- For $A \in \mathbb{C}^{n\times n}$, we denote by μ_A its empirical spectral distribution:

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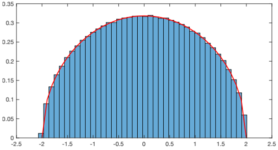

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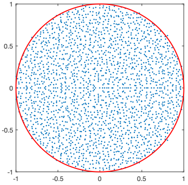

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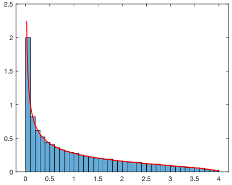

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- *Is there an "entropy" which is monotonic along these theorems?*

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Derivation of the formula

- Let $\nu \in \mathcal{P}(\mathbb{R})$ be some spectral measure (macrostate).

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- We find that

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For details: Biane, *Entropie libre et algèbres d'opérateurs* (2002).

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For details: Ben Arous–Guionnet (1997), Hiai–Petz (2000).

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Variational characterizations of the limiting distributions

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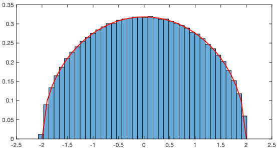

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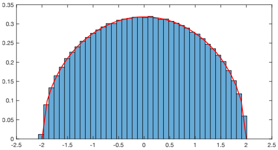

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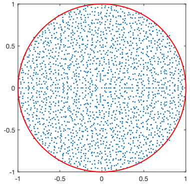

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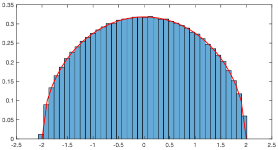

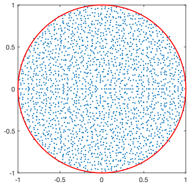

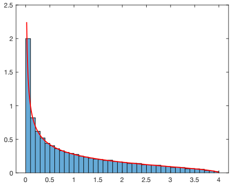

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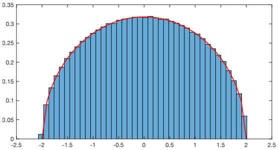

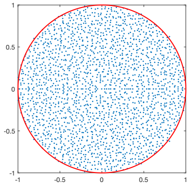

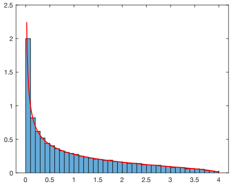

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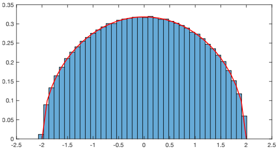

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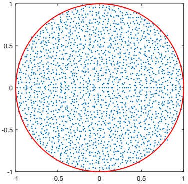

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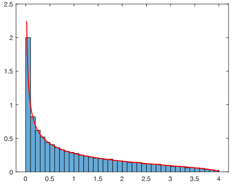

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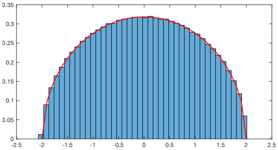

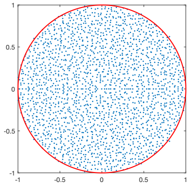

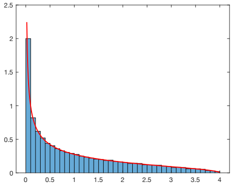

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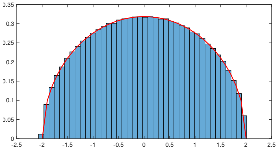

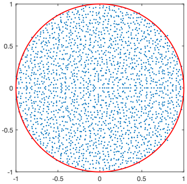

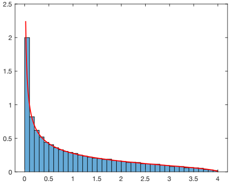

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For details: Saff–Totik (1996).

Monotonicity of $\mathcal{E}$ in the unitary Gaussian case
(Chafaï–D.–Youssef, 2022)

Suppose $X_n := (\xi_{ij})_{1\leq i,j\leq n}$ are i.i.d. **standard complex Gaussian**.

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$$\mathcal{E}^{(\mathrm{W})}(\mathbb{E}\mu_{W_n}) =$$

Theorem 2

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Theorem 3

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(Chafaï–D.–Youssef, 2022)

Suppose $X_n := (\xi_{ij})_{1\le i,j\le n}$ are i.i.d. **standard complex Gaussian**.

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$$\mathcal{E}^{(\mathrm{W})}(\mathbb{E}\mu_{W_n}) = \frac{3}{4} + \frac{1}{2}\left(\log n + \gamma + \frac{1}{2n} - H_n\right),$$

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Monotonicity of $\mathcal{E}$ in the unitary Gaussian case

Summarizing plots

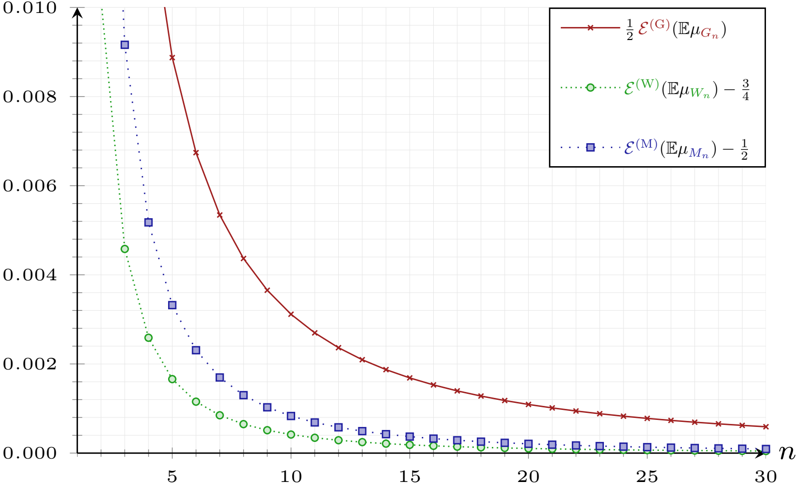

Figure: Plots of the logarithmic energies in the unitary Gaussian case.

Two open questions

Recall that $X_n := (\xi_{ij})_{1 \leq i,j \leq n}$ is i.i.d. with the $\mathcal{N}_{\mathbb{C}}(0,1)$ law and

$$G_n := \frac{1}{\sqrt{n}} X_n, \quad W_n := \frac{G_n + G_n^*}{\sqrt{2}}, \quad M_n := G_n G_n^*.$$

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Question. What is the meaning of that series?

Also,

$$\mathcal{E}^{(M)}(\mathbb{E}\mu_{M_n}) = 2\,\mathcal{E}^{(W)}(\mathbb{E}\mu_{W_n}).$$

Question. Is there a quick, conceptual proof of this fact?

Some empirical results

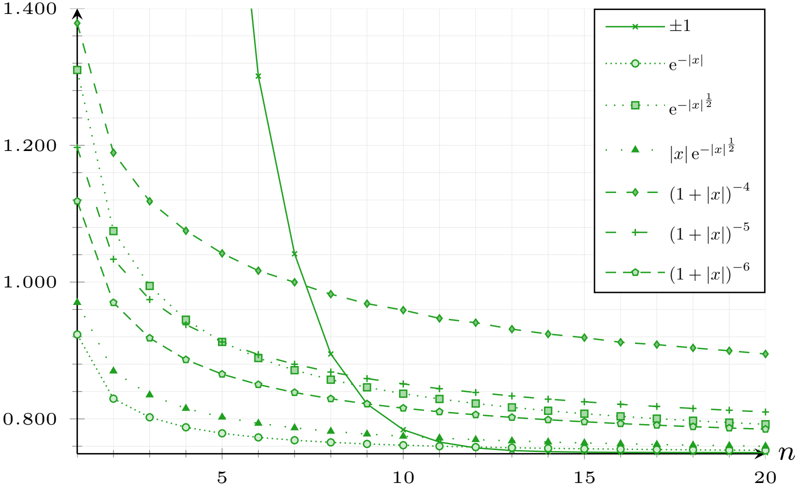

Figure: Simulation of $\mathcal{E}^{(W)}(\mathbb{E}\mu_{W_n})$ for different distributions of $X_n := (\xi_{ij})$.

Some empirical results, II

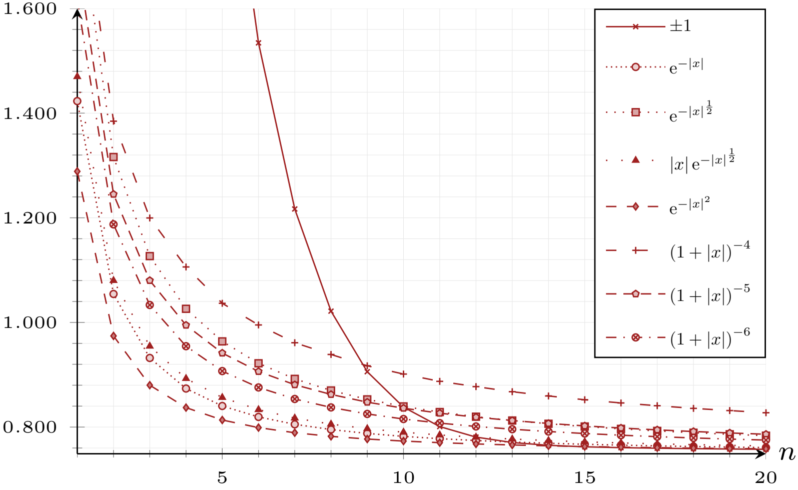

Figure: Simulation of $\mathcal{E}^{(\mathsf{G})}(\mathbb{E}\mu_{G_n})$ for different distributions of $X_n := (\xi_{ij})$.

Some empirical results, III

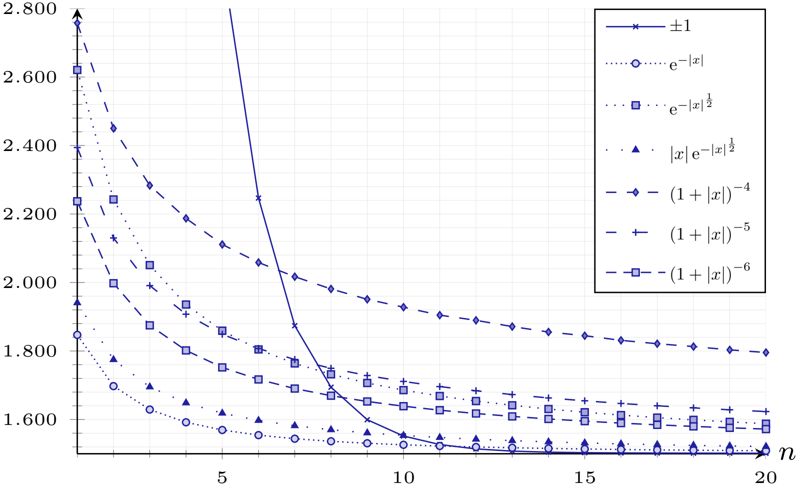

Figure: Simulation of $\mathcal{E}^{(\mathsf{M})}(\mathbb{E}\mu_{M_n})$ for different distributions of $X_n \coloneqq (\xi_{ij})$.

Some empirical results, IV

Gaussian β-ensemble: $f_n^{(\beta)}(\Lambda) \propto e^{-\frac{2+\beta(n-1)}{4}\sum_{i=1}^n \lambda_i^2} \prod_{i<j} |\lambda_i - \lambda_j|^\beta$ and

$$\mu_n^{(\beta)}(\mathrm{d}x) := \int \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i}(\mathrm{d}x)\, f(\Lambda)\mathrm{d}\Lambda.$$

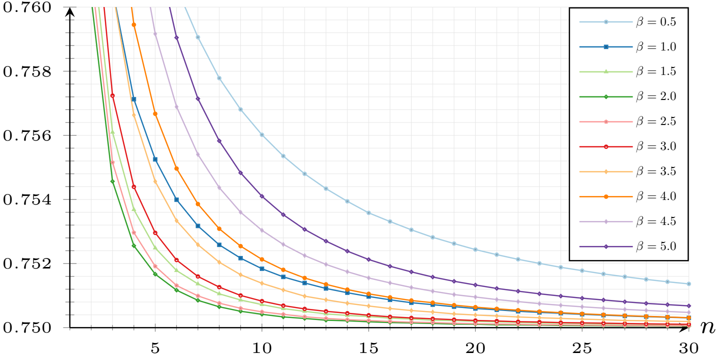

Figure: Simulation of $\mathcal{E}^{(\mathrm{W})}(\mu_n^{(\beta)})$ for different values of β.

Level densities of Ginibre, GUE, and LUE

- $\mathbb{E}\mu_{G_n}(\mathrm{d}z) = \sqrt{n}\,\sigma_n^{\mathrm{Gin}}(z\sqrt{n})\,\mathrm{d}z$ with

$$\sigma_n^{\mathrm{Gin}}(z) := \frac{\mathrm{e}^{-|z|^2}}{n\pi}\sum_{k=0}^{n-1}\frac{|z|^{2k}}{k!}, \qquad z \in \mathbb{C};$$

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$$\sigma_n^{\mathrm{GUE}}(x) := \frac{\mathrm{e}^{-x^2}}{n\sqrt{\pi}}\sum_{k=0}^{n-1}\frac{h_k(x)^2}{2^k\,k!}, \qquad x \in \mathbb{R},$$

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- $\mathbb{E}\mu_{M_n}(\mathrm{d}s) = n\,\sigma_n^{\mathrm{LUE}}(ns)\,\mathrm{d}s$ with

$$\sigma_n^{\mathrm{LUE}}(s) := \frac{\mathrm{e}^{-s}}{n}\sum_{k=0}^{n-1}L_k(s)^2, \qquad s \geq 0,$$

  where L_k, $k \geq 0$, are the Laguerre polynomials.

For details: Mehta, *Random matrices* (2004).

Exponential weight formulation of the logarithmic energy

Lemma

Let ν be a probability measure such that $\int \log(1+|x|)\,\nu(\mathrm{d}x) < \infty$. Then for any $p > 0$,

$$-\Sigma(\nu) = \frac{\gamma}{p} - \frac{1}{p}\int_0^\infty \frac{\mathrm{d}t}{t}\left(\frac{1}{1+t} - \iint \mathrm{e}^{-t|x-y|^p}\,\nu(\mathrm{d}x)\nu(\mathrm{d}y)\right).$$

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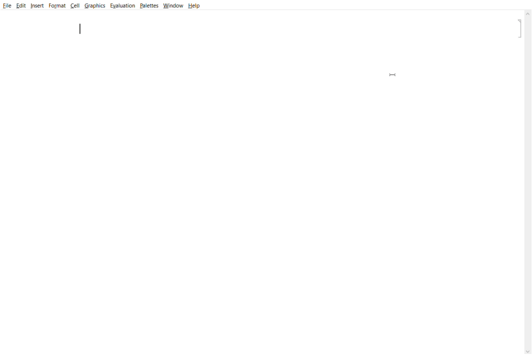
- $p = 2$ suited for the Gaussian kernel e^{-x^2} (GUE);
- $p = 1$ suited for the Laguerre kernel e^{-s} (LUE).

Evaluating binomial sums: an example

Lemma

Let $O_n := 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}$. *Then*

$$16 \sum_{0 \le k < \ell \le n} (-1)^{k+\ell} \binom{n-1}{k} \binom{k-\frac{1}{2}}{n} \binom{n-1}{\ell} \binom{\ell-\frac{1}{2}}{n} O_{\ell-k} = \qquad (?)$$



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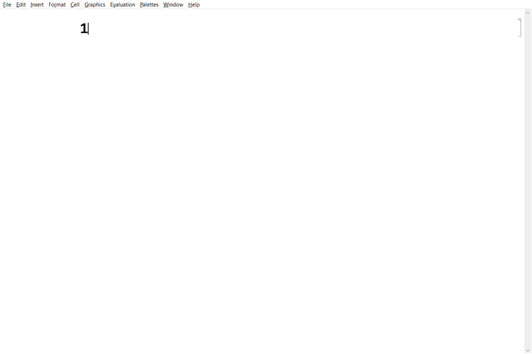

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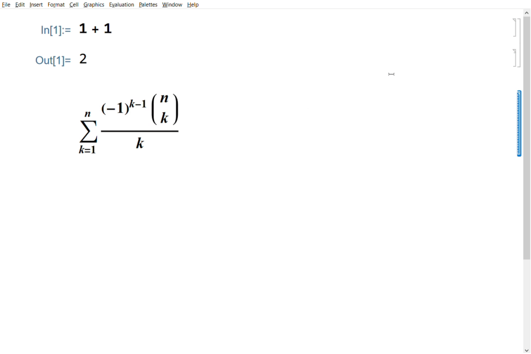

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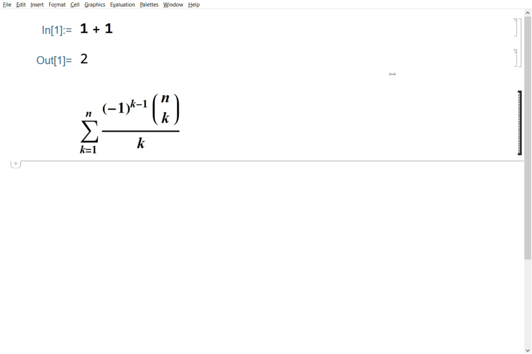

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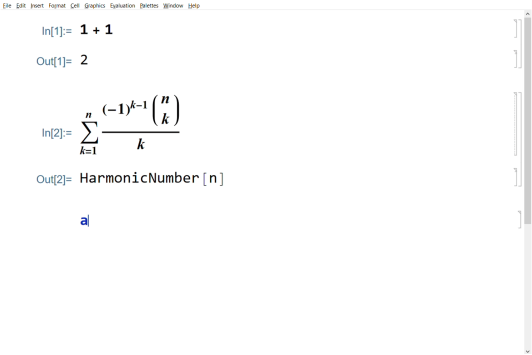

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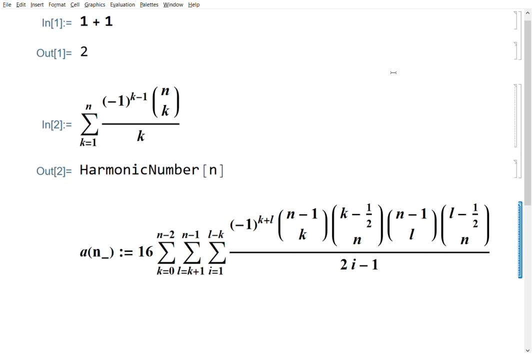

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In[1]:= 1 + 1

Out[1]= 2

In[2]:= $\sum_{k=1}^{n} \frac{(-1)^{k-1}\binom{n}{k}}{k}$

Out[2]= HarmonicNumber[n]

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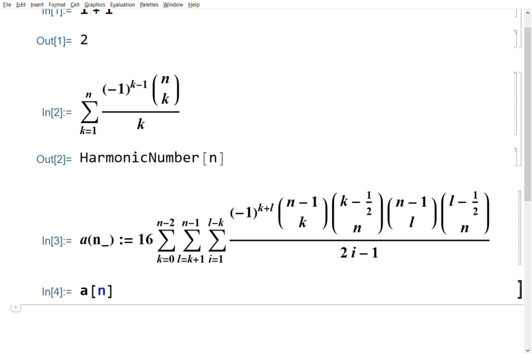

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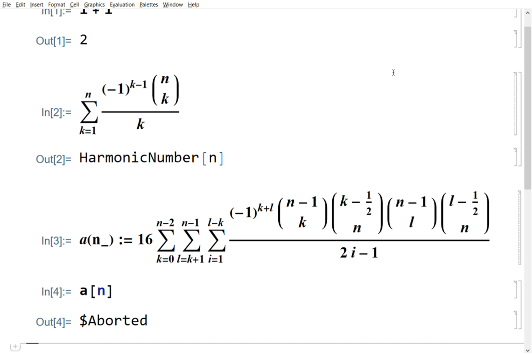

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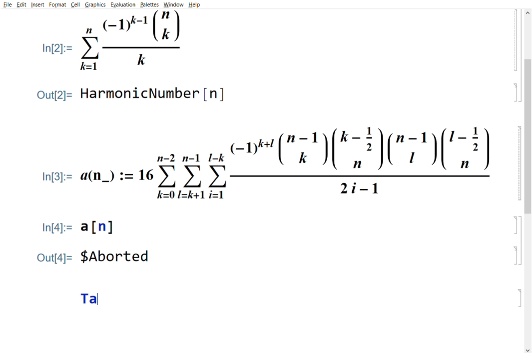

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In[4]:= `a[n]`

Out[4]= `$Aborted`

In[5]:= `Table[a[n], {n, 1, 10}]`

Out[5]= $\left\{0, \frac{3}{4}, \frac{7}{6}, \frac{35}{24}, \frac{101}{60}, \frac{28}{15}, \frac{283}{140}, \frac{1207}{560}, \frac{5729}{2520}, \frac{1199}{504}\right\}$

F|

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```
In[6]:= FindSequenceFunction[%][n]
```

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In[6]:= `FindSequenceFunction[%][n]`

Out[6]= $\frac{1 - n + 2\,\text{EulerGamma}\,n + 2\,n\,\text{PolyGamma}[0, n]}{2\,n}$

Thanks for your attention!