THE SUPPORT FUNCTION OF THE HIGH-DIMENSIONAL POISSON POLYTOPE

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Abstract. We study the asymptotic behavior as $d \to \infty$ of the support function

$$h_\lambda^d(u) := \sup_{x \in K_\lambda^d} \langle u, x \rangle$$

in an arbitrary direction $u \in \mathbb{S}^{d-1}$ of the Poisson polytope K_λ^d sampled in the Euclidean ball $\mathbb{B}^d$. We identify three different regimes (subcritical, critical, and supercritical) in terms of the intensity $\lambda := \lambda(d)$ of the underlying Poisson process and derive in each regime the precise distributional convergence of h_λ^d after suitable scaling. We also treat the same question where the support function is considered over multiple directions at once. We finally deduce weak counterparts for the radius-vector function of the polytope.

Keywords. Support function; random convex polytopes; phase transition in high dimension.

1. Introduction

The study of high-dimensional polytopes has attracted recent attention in stochastic geometry [4, 16, 22, 15]. This is notably due to the fact that several classical models of random polytopes can be defined in any dimension and as such, they provide natural examples which might confirm or deny several of the famous conjectures from high-dimensional convex geometry. For example, the work [17] deals with Poisson polytopes generated by random hyperplane tessellations in the context of the hyperplane conjecture.

In this paper, we consider the Poisson polytope K_λ^d obtained as the convex hull of a Poisson point process with intensity $\lambda := \lambda(d)$ sampled in the Euclidean, d-dimensional unit ball $\mathbb{B}^d$. Together with its binomial variant obtained when replacing the Poisson point process with a fixed number n of i.i.d. variables uniformly distributed in $\mathbb{B}^d$, this model of random polytope has proved to be one of the most intensively studied in the literature. In particular, several non-asymptotic results are known, including Wendel's calculation [30] of the probability that the origin belongs to the random polytope, Efron and Buchta's mean-value identities [12, 6] and the recent work due to Kabluchko [20, 21] which provides an explicit formula for the expectation of the f-vector constituted with the number of k-dimensional facets of K_λ^d. The asymptotic study of K_λ^d, when the dimension is fixed and the intensity or the number of input points is large, dates back to the seminal work due to Rényi and Sulanke in dimension two [24, 25] and has been then carried through many subsequent works which have made explicit limit expectations for several functionals [27, 28, 23], variance bounds and limit variances [2, 8] and functional limit theorems for the support function and radius-vector function [7] when the enclosing convex body is a ball.

In the continuation of the early work of Bárány and Füredi [1], who investigated the phase transition for the probability of having all sampled points in convex position, the high-dimensional study of K_λ^d and variants has been continued over the last decades [11, 13, 4, 9, 5], with more emphasis on the threshold

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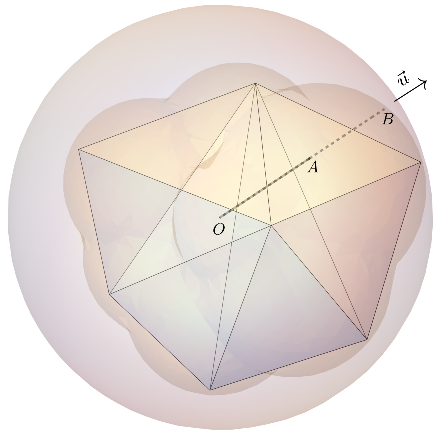

FIGURE 1. The support function $h_\lambda^d(u) := OB$ and the radius-vector function $\rho_\lambda^d(u) := OA$ of the polytope K_λ^d in dimension $d = 3$. The Voronoi flower is the union of the closed balls $(\frac{x}{2} + \frac{|x|}{2}\mathbb{B}^d)$ where x runs over the vertices of K_λ^d.

for the emergence of significant volumes. Notably, in [5], Bonnet, Kabluchko and Turchi estimated the asymptotics for different intensity regimes of the mean volume of a more general polytope called the β-polytope which includes the case of K_λ^d (save for the fact that they consider a binomial version of it). In particular, they exhibit a critical phase when the logarithm of the number of input points is comparable to $\frac{d}{2}\log d$, i.e., they show that the expected normalized volume of the polytope vanishes when the number of input points increases slower than $d^{d/2}$, and persists when this number increases faster than $d^{d/2}$. At the critical threshold, when the number grows like $(d/2x)^{d/2}$, they prove a convergence to e^{-x}.

Associated with the (random) polytope K_λ^d are the random processes h_λ^d and ρ_λ^d, respectively called the *support function* and the *radius-vector function*, given for any direction u in the unit sphere $\mathbb{S}^{d-1}$ by

$$h_\lambda^d(u) := \sup\left\{\langle u, x\rangle : x \in K_\lambda^d\right\}, \tag{1}$$

and

$$\rho_\lambda^d(u) := \sup\left\{t > 0 : tu \in K_\lambda^d\right\}. \tag{2}$$

The support function of K_λ^d corresponds to the radius-vector function of its Voronoi flower, see Figure 1. Even though h_λ^d and ρ_λ^d are one-dimensional statistics, they are known to fully characterize the convex body [26, Theorem 1.7.1], so understanding h_λ^d or ρ_λ^d as $d \to \infty$ already sheds significant light on the asymptotic geometry of K_λ^d. It appears however that the analysis of the radius-vector function is more delicate and we mostly focus on the support function in this work.

The distribution of the random variables $h_\lambda^d(u)$ and $\rho_\lambda^d(u)$ does not depend on u by rotational invariance; we let $h_\lambda^d := h_\lambda^d(u)$ and $\rho_\lambda^d := \rho_\lambda^d(u)$, and consider that u is either fixed in $\mathbb{S}^{d-1}$ or chosen uniformly at random in $\mathbb{S}^{d-1}$ (independently of the Poisson process). Given the following technical assumption on λ,

$$\liminf_{d\to\infty} \frac{\lambda\kappa_d}{d} > 2, \tag{H}$$

where $\kappa_d := |\mathbb{B}^d|$ is the d-dimensional volume of the unit ball, the random variables h^d_λ and ρ^d_λ will take values in $[0,1]$ with high probability as $d \to \infty$ (see Lemma 2.1). This hypothesis means that $\lambda\kappa_d$, i.e., the mean number of points in $\mathbb{B}^d$ of the Poisson process, is asymptotically greater than twice the dimension d.

Our first result identifies the *subcritical*, *critical*, and *supercritical regimes* where h^d_λ tends either to 0, to a value in $(0,1)$, or to 1 as $d \to \infty$, depending on whether $\log \lambda\kappa_d$ grows slower than, comparably to, or faster than the dimension d.

Theorem 1.1 (Asymptotic regimes of h^d_λ)**.** *Under Assumption (H), suppose that*

$$x := \lim_{d\to\infty} \frac{1}{d} \log \lambda\kappa_d$$

exists in $[0,\infty]$. Then the following holds in probability as $d \to \infty$:

$$\begin{cases} h^d_\lambda \sim \sqrt{\dfrac{2}{d}\log \lambda\kappa_d}, & \textit{if } x = 0, \\ h^d_\lambda \to \sqrt{1-\mathrm{e}^{-2x}}, & \textit{if } x \in (0,\infty), \\ 1 - h^d_\lambda \sim \dfrac{1}{2}(\lambda\kappa_d)^{-\frac{2}{d+1}}, & \textit{if } x = \infty. \end{cases}$$

Note that Assumption (H) is automatically fulfilled in the critical and supercritical regime ($x \in (0,\infty]$).

In addition to Theorem 1.1, we can obtain the distributional fluctuations of h^d_λ around its limiting value. Moreover, we can do the same when the support function is considered over several directions at once. Namely, we let $m \geq 1$ be a fixed integer and define

$$h^{d,m}_\lambda \quad := \inf_{u \in \mathbb{S}^{d-1}\cap\mathbb{R}^m} h^d_\lambda(u).$$

Again, by rotational invariance, the distribution of this infimum does not depend on the direction of the linear m-dimensional section of $\mathbb{S}^{d-1}$, and we have $h^{d,1}_\lambda \overset{(d)}{=} h^d_\lambda$. In particular, we expect $h^{d,m}_\lambda$ to behave like h^d_λ, i.e., satisfy the exact same conclusions of Theorem 1.1 with the same threshold. This is confirmed by Theorem 1.2 below which provides a Gumbel limit distribution for a functional of $h^{d,m}_\lambda$ in the case $m = 1$ when Assumption (H) is fulfilled and in the case $m \geq 2$ when $\log \lambda\kappa_d$ belongs to the asymptotic range $\left(\log d, d^{\frac{3}{2}}\right)$.

Theorem 1.2 (Distributional limit)**.** *Let $m \geq 1$ be a fixed integer, and let $\lambda := \lambda(d) > 0$ satisfy Assumption (H) and either one of the three assumptions* ($\mathrm{A}_{\mathrm{sub}}$)*,* ($\mathrm{A}_{\mathrm{crit}}$)*, or* ($\mathrm{A}_{\mathrm{sup}}$) *given in Lemma 4.4 when $m \geq 2$. Then there exist two explicit sequences $\mathfrak{a}(d;m)$ and $\mathfrak{b}(d;m)$ (given at* (38) *and* (39) *for $m \geq 2$, and at* (10) *and* (11) *for $m = 1$) such that*

$$\mathfrak{a}(d;m) - \mathfrak{b}(d;m) \log \frac{1}{\sqrt{1-\left(h^{d,m}_\lambda\right)^2}}$$

converges in law as $d \to \infty$ towards the standard Gumbel distribution.

Theorem 1.1 is a direct consequence of Theorem 1.2. In particular, the statement of Theorem 1.1 can be extended to $h^{d,m}_\lambda$ in the case $m \geq 2$.

The proof of Theorem 1.2 for $m \geq 2$ relies essentially on the *ad hoc* application of a remarkable result due to S. Janson on random coverings of a set [18].

Theorem 1.1 above is clearly reminiscent of [5, Theorem 3.1]. Comparing both results, we observe that if λ is taken so that $\log \lambda\kappa_d$ belongs to the asymptotic range $[d, d\log d)$, then the expected volume ratio $\mathbb{E}\,|K_\lambda^d|/\kappa_d$ vanishes as observed in [5], while according to Theorem 1.2, K_λ^d still has long 'arms' in any finite number of independent directions. This confirms the well-known picture of a high-dimensional convex body which was popularized by Vitali Milman and which looks like a 'star-shaped body with a lot of points very far from the origin and lot of points very close to the origin' [14, Section 2]. Studying the minimum of the support function over a section of K_λ^d may provide a way of quantifying the size of the 'holes', i.e., estimating the critical dimension under which the support function is close to one in every direction of the section of K_λ^d and above which we expect to see directions almost unoccupied by the section of K_λ^d. Unsurprisingly, Theorem 1.2 suggests that as soon as we reach the threshold for the one-dimensional section given in Theorem 1.1, we expect every section with fixed dimension m to look like the m-dimensional unit ball. The rest of the study should then lead us to consider m tend to infinity with d in the supercritical case of Theorem 1.1 in order to decide when exactly the function $h_\lambda^{d,m}$ switches from being almost equal to 1 to being almost equal to 0. This requires a serious revision of the covering methods used in the proof of Theorem 1.2 that we leave for further work.

The paper is structured as follows. We start in Section 2 with some asymptotic preliminaries, notably for the incomplete beta function to which the tail probability of h_λ^d and $h_\lambda^{d,m}$ is related. As a warm-up, in Section 3, we establish Theorem 1.2 in the case $m = 1$ by elementary means, see Theorem 3.2, and deduce from it Theorem 1.1. Section 4 is devoted to the use of Janson's covering techniques for proving Theorem 1.2 in the case $m \geq 2$. Finally, in Section 5, we transfer some of our results to the radius-vector function ρ_λ^d of K_λ^d.

Notation. All asymptotic estimates are w.r.t. the dimension d: $g \gg f$ (or $f \ll g$, or $f = o(g)$) for nonnegative sequences $f := f(d)$ and $g := g(d)$ means that $f(d) \leq \varepsilon g(d)$ holds for all d sufficiently large and any $\varepsilon > 0$, while $f = O(g)$ or $f \lesssim g$ indicates that $f(d) \leq Cg(d)$ holds for all $d \geq 1$ and some constant $C > 0$. We also write $f \sim g$ if $|f - g| \ll g$, and $f \asymp g$ if $f \lesssim g$ and $g \lesssim f$.

2. Preliminaries

This section aims at providing an explicit formula for the distribution function of h_λ^d in terms of the incomplete beta function. It also paves the way for the asymptotic study of the tail probability of h_λ^d and $h_\lambda^{d,m}$, which is the focus of Sections 3 and 4.

Let $\mathcal{P}_\lambda^d$, $d \geq 2$, be Poisson point processes (embedded in a common abstract probability space $(\Omega, \mathcal{A}, \mathbb{P})$) with intensities $\lambda := \lambda(d) > 0$ in $\mathbb{R}^d$. The polytope K_λ^d is defined as the convex hull of $\mathcal{P}_\lambda^d \cap \mathbb{B}^d$, where $\mathbb{B}^d := \{x \in \mathbb{R}^d : |x| \leq 1\}$ is the Euclidean unit ball of $(\mathbb{R}^d, \|\cdot\|)$, with the Euclidean norm $\|\cdot\|$ derived from the usual inner product $\langle \cdot, \cdot \rangle$. We recall that $|\cdot|$ denotes the d-dimensional Lebesgue measure of $\mathbb{R}^d$. In both notations $\|\cdot\|$ and $|\cdot|$, the dependency on d is implicit.

A priori, the support function h_λ^d introduced in (1) takes values in $\{-\infty\} \cup (-1, 1)$, but if λ is not chosen too small, then the polytope K_λ^d will likely contain the origin, which means that $h_\lambda^d \geq 0$.

Lemma 2.1 (Containing the origin)**.** *If Assumption (H) holds, then* $\mathbb{P}(0 \in K_\lambda^d) \to 1$ *as* $d \to \infty$*. If however* $\limsup_{d\to\infty} \frac{\lambda\kappa_d}{d} < 2$*, then* $\mathbb{P}(0 \in K_\lambda^d) \to 0$ *as* $d \to \infty$*.*

Proof of Lemma 2.1. The number $N := N(d)$ of points in K_λ^d has a Poisson distribution with mean $\lambda\kappa_d$. Further, conditional on N, those points have a symmetric distribution in $\mathbb{R}^d$. It follows from Wendel's formula [30] that

$$\begin{aligned}\mathbb{P}\big(0 \notin K_\lambda^d \mid N\big) &= \mathbb{1}_{\{N\le d\}} + \mathbb{1}_{\{N>d\}}\, 2^{-(N-1)} \sum_{k=0}^{d-1} \binom{N-1}{k} \\ &= 1 - \mathbb{1}_{\{N>d\}}\, 2^{-(N-1)} \sum_{k=d}^{N-1} \binom{N-1}{k} \\ &= 1 - \mathbb{P}(S_{N-1} \ge d \mid N),\end{aligned}$$

where S_n, $n \ge 0$, are Binomial$(n, \frac{1}{2})$ random variables independent of N. Hence

$$\mathbb{P}\big(0 \in K_\lambda^d\big) = \mathbb{P}(S_{N-1} \ge d).$$

By the law of large numbers, $\mathbb{P}(S_{N-1} \ge d) \to 1$ as $d \to \infty$ if

$$\liminf_{d\to\infty} \frac{1}{d}\, \mathbb{E}[S_{N-1}] > 1,$$

that is (since $\mathbb{E}[S_{N-1}] = \frac{\lambda\kappa_d - 1}{2}$), if $\lambda\kappa_d \ge (2+\varepsilon)d$ holds for some $\varepsilon > 0$ and all d sufficiently large; similarly, $\mathbb{P}(S_{N-1} \ge d) \to 0$ if $\lambda\kappa_d \le (2-\varepsilon)d$ holds for some $\varepsilon > 0$ and all d sufficiently large. (In fact, by classical large deviation theory, these two convergences occur exponentially fast.) $\square$

Taking Assumption (H) for granted, we thus have $0 \le h_\lambda^d \le 1$ w.h.p. as $d \to \infty$. Now,

$$\mathbb{P}\Big(h_\lambda^d \le r\Big) = \mathbb{P}\Big(h_\lambda^d(u) \le r\Big), \quad 0 \le r \le 1,$$

for, e.g., $u := (1, 0, \ldots) \in \mathbb{S}^{d-1}$; we compute this probability as

$$\mathbb{P}\Big(\mathcal{P}_\lambda \cap \mathcal{C}^d(r; u) = \emptyset\Big) = \mathrm{e}^{-\lambda|\mathcal{C}^d(r;u)|},$$

where the spherical cap $\mathcal{C}^d(r; u) := \{x \in \mathbb{B}^d : \langle u, x\rangle > r\}$ has volume

$$|\mathcal{C}^d(r; u)| = \kappa_{d-1} \int_r^1 (1-t^2)^{\frac{d-1}{2}}\, \mathrm{d}t = \frac{\kappa_{d-1}}{2} \int_0^{1-r^2} v^{\frac{d-1}{2}} (1-v)^{-\frac{1}{2}}\, \mathrm{d}v, \tag{3}$$

with the last integral resulting from the change of variable $v \leftarrow 1 - t^2$. Hence

$$\mathbb{P}\Big(h_\lambda^d \le r\Big) = \exp\Big(-\frac{\lambda\kappa_{d-1}}{2}\, \mathrm{B}\big(1 - r^2; \tfrac{d+1}{2}, \tfrac{1}{2}\big)\Big), \tag{4}$$

where

$$\mathrm{B}(x; p, q) := \int_0^x v^{p-1}(1-v)^{q-1}\, \mathrm{d}v, \quad x \in [0, 1],\ p, q > 0,$$

is the *lower incomplete beta function* (the complete beta function is $\mathrm{B}(p, q) := \mathrm{B}(1; p, q)$). We will see in Section 4 that when considering the support function over $m \ge 2$ directions at once, the distribution function of the infimum $h_\lambda^{d,m}$ also involves this special function (with the third parameter $q = \frac{1}{2}$ replaced by $q = \frac{m}{2}$) as well as the volume of unit balls. Thus, the asymptotic behavior of h_λ^d and of $h_\lambda^{d,m}$ will depend on the interplay between the intensity $\lambda := \lambda(d)$ and the two quantities κ_n and $\mathrm{B}(x; p, q)$, where n, x and p may depend on the dimension d.

Regarding κ_d, we will essentially use the asymptotic relation

$$\kappa_{d-1} = \kappa_d \sqrt{\frac{d}{2\pi}} \left[1 + O\left(\frac{1}{d}\right)\right], \tag{5}$$

which is easily derived from the well-known formula

$$\kappa_d = \frac{\pi^{\frac{d}{2}}}{\Gamma\left(1 + \frac{d}{2}\right)} \tag{6}$$

and Stirling's formula for Euler's gamma function Γ (see, e.g., [29, (3.24)]). As for the incomplete beta function, we can derive basic first-order estimates:

Lemma 2.2 (Estimates of the incomplete beta function). *Let $p, q > 0$. Then for any $x \in (0,1)$ such that $(p+q)x < p+1$,*

$$\text{the ratio} \quad \mathrm{B}(x;p,q) \Big/ \frac{x^p(1-x)^{q-1}}{p} \quad \text{lies between } 1 \text{ and } \frac{1}{1 - \frac{q-1}{p+1} \cdot \frac{x}{1-x}}.$$

In particular, when $x \in (0,1)$ and $p, q > 0$ are three sequences indexed by d:

1. *If $p \gg \dfrac{|q-1|x}{1-x}$, then*
$$\mathrm{B}(x;p,q) = \frac{x^p(1-x)^{q-1}}{p}\left[1 + O\left(\frac{|q-1|x}{p(1-x)}\right)\right].$$
2. *If $p \gg \dfrac{|q-1|}{1-x}$, then*
$$\frac{\mathrm{B}(x;p,q)}{\mathrm{B}(p,q)} = \frac{x^p[(1-x)p]^{q-1}}{\Gamma(q)}\left[1 + O\left(\frac{|q-1|}{p(1-x)}\right)\right].$$

Proof. For $x \in (0,1)$ we have from [29, (11.33)] the series representation

$$B(x;p,q) = \frac{x^p(1-x)^q}{p} \sum_{n=0}^{\infty} \frac{(p+q)_n}{(p+1)_n} x^n,$$

where the ratio of Pochhammer symbols

$$\frac{(p+q)_n}{(p+1)_n} := \frac{p+q}{p+1} \cdot \frac{p+q+1}{p+2} \cdots \frac{p+q+n-1}{p+n}$$

belongs to $\left[1, \left(\frac{p+q}{p+1}\right)^n\right]$ if $q \geq 1$, and to $\left[\left(\frac{p+q}{p+1}\right)^n, 1\right]$ if $0 < q < 1$. Now if $(p+q)x < p+1$, then

$$\sum_{n=0}^{\infty} \left(\frac{p+q}{p+1}\right)^n x^n = \frac{1}{1 - \frac{p+q}{p+1}x} = \frac{1}{1-x} \cdot \frac{1}{1 - \frac{q-1}{p+1} \cdot \frac{x}{1-x}},$$

hence the lower and upper bounds on $\mathrm{B}(x;p,q)/\frac{x^p(1-x)^{q-1}}{p}$. The first stated asymptotic estimate is an immediate consequence of these bounds, while the second one follows from the first one combined with

$$\mathrm{B}(p,q) = \Gamma(q) \cdot \frac{\Gamma(p)}{\Gamma(p+q)} = \frac{\Gamma(q)}{p^q}\left[1 + O\left(\frac{|q-1|}{p}\right)\right],$$

see, e.g., [29, (3.31)]. □

3. The support function in one direction

We are now ready to establish Theorem 1.1 as well as Theorem 1.2 in the case $m = 1$: we do so by recalling the distribution function (4) of the support function h_λ^d, then plug in asymptotics for the incomplete beta function (Lemma 2.2) and for the volume of the Euclidean unit balls (5).
To start with, we identify the limit of h_λ^d in probability in Lemma 3.1. We then derive in Theorem 3.2 a convergence in distribution for a proper renormalized version of h_λ^d. We finally use Theorem 3.2 to prove Theorem 1.1 as well as Theorem 1.2 in the case $m = 1$.

Lemma 3.1 (Limit in probability of the support function)**.** *Under the assumption of Theorem 1.1,*

$$\lim_{d\to\infty} h_\lambda^d = \sqrt{1-\mathrm{e}^{-2x}}, \quad \textit{in probability}$$

(with the convention $\mathrm{e}^{-\infty} = 0$*).*

Proof. We start by applying Lemma 2.2 with only the second argument of the incomplete beta function depending on d: for any fixed $r \in (0,1)$,

$$\mathrm{B}\big(1-r^2; \tfrac{d+1}{2}, \tfrac{1}{2}\big) = \frac{2(1-r^2)^{\frac{d+1}{2}}}{rd}\left[1 + O\left(\frac{1}{d}\right)\right].$$

Inserting this and the other estimate (5) into (4) then yields

$$\begin{aligned} -\log \mathbb{P}\Big(h_\lambda^d \le r\Big) &= \frac{\lambda\kappa_d(1-r^2)^{\frac{d+1}{2}}}{r\sqrt{2\pi d}}\left[1 + O\left(\frac{1}{d}\right)\right] \\ &= \exp\left\{\log\lambda\kappa_d + \frac{d+1}{2}\log(1-r^2) - \log r\sqrt{2\pi d} + O\left(\frac{1}{d}\right)\right\} \qquad (7) \\ &= \exp\left\{d\left(\frac{\log\lambda\kappa_d}{d} + \log\sqrt{1-r^2} + o(1)\right)\right\}. \end{aligned}$$

Since $\frac{\log\lambda\kappa_d}{d} \to x \in [0,\infty]$ as $d \to \infty$, the change of sign in this exponent provides the required threshold, i.e.,

$$\lim_{d\to\infty} \mathbb{P}\Big(h_\lambda^d \le r\Big) = \begin{cases} 0 & \text{if } r < \sqrt{1-\mathrm{e}^{-2x}}, \\ 1 & \text{if } r > \sqrt{1-\mathrm{e}^{-2x}}. \end{cases}$$

This proves the convergence in distribution of h_λ^d towards the constant $\sqrt{1-\mathrm{e}^{-2x}}$, which is equivalent to the convergence in probability to the same limit. □

By also letting the first argument of the incomplete beta function depend on d, a deeper application of Lemma 2.2 enables us to complete the proof of Theorems 1.1 and 1.2 in the case $m = 1$.

Theorem 3.2 (Convergence in distribution of the renormalized support function)**.** *Under the assumption of Theorem 1.1, the random variable*

$$d\left(\log\frac{1}{\sqrt{1-(h_\lambda^d)^2}} - \frac{1}{d+1}\log\lambda\kappa_d\right) + \log\sqrt{\mathfrak{m}(d)}$$

converges towards the standard Gumbel distribution as $d \to \infty$, where

$$\mathfrak{m}(d) := \begin{cases} 4\pi \log \lambda\kappa_d, & \textit{in the subcritical regime } \log \lambda\kappa_d \ll d, \\ 2\pi d(1-\mathrm{e}^{-2x}), & \textit{in the critical regime } \log \lambda\kappa_d \sim dx \textit{ with } x \in (0,\infty), \\ 2\pi d\Big(1-(\lambda\kappa_d)^{-\frac{2}{d+1}}\Big), & \textit{in the supercritical regime } \log \lambda\kappa_d \gg d. \end{cases}$$

Remark 3.3. *In the critical regime where $\log \lambda\kappa_d = dx + y + o(1)$ with $x \in (0,\infty)$ and $y \in \mathbb{R}$, the theorem reads*

$$d\left(\log \frac{1}{\sqrt{1-(h_\lambda^d)^2}} - x\right) + \log\sqrt{2\pi d(1-\mathrm{e}^{-2x})} \underset{d\to\infty}{\Longrightarrow} G + y,$$

that is,

$$\sqrt{2\pi d}\,\mathrm{e}^{-dx}\left(1-(h_\lambda^d)^2\right)^{-\frac{d}{2}} \underset{d\to\infty}{\Longrightarrow} \frac{\mathrm{e}^{G+y}}{\sqrt{1-\mathrm{e}^{-2x}}},$$

for some standard Gumbel variable G.

Proof. We apply Lemma 2.2 again by letting $r \in (0,1)$ in the previous proof depend on d. If $dr^2 \gg 1$ holds, then as a replacement of (7) we may write

$$-\log \mathbb{P}\Big(h_\lambda^d \le r\Big) = \exp\left\{\log \lambda\kappa_d + \frac{d+1}{2}\log(1-r^2) - \log r\sqrt{2\pi d} + o(1)\right\}. \tag{8}$$

Fix $\tau \in \mathbb{R}$ and choose $r := r(d;\tau)$ as the (asymptotically unique) solution to the equation

$$\log \lambda\kappa_d + \frac{d+1}{2}\log(1-r^2) - \log r\sqrt{2\pi d} = -\tau, \quad \text{namely} \quad \frac{(1-r^2)^{\frac{d+1}{2}}}{r} = \frac{\sqrt{2\pi d}}{\lambda\kappa_d}\,\mathrm{e}^{-\tau}. \tag{9}$$

In particular, under Assumption (H),

$$-\frac{d+1}{2}\log(1-r^2) \ge \log\frac{\lambda\kappa_d}{\sqrt{2\pi d}} + \tau \to \infty,$$

so that, indeed, $dr^2 \gg 1$ and (8) is true. Inserting (9) there then yields

$$\lim_{d\to\infty} \mathbb{P}\Big(h_\lambda^d \le r\Big) = \mathrm{e}^{-\mathrm{e}^{-\tau}},$$

which is the c.d.f. of the standard Gumbel distribution. Now, we observe that

$$\begin{aligned} \mathbb{P}\Big(h_\lambda^d \le r\Big) &= \mathbb{P}\left[-\frac{d+1}{2}\log\Big(1-(h_\lambda^d)^2\Big) - \log\frac{\lambda\kappa_d}{r\sqrt{2\pi d}} \le \tau\right] \\ &= \mathbb{P}\left[d\left(\log\frac{1}{\sqrt{1-(h_\lambda^d)^2}} - \frac{1}{d+1}\log\lambda\kappa_d\right) + \frac{\log r\sqrt{2\pi d}}{1+O(\frac{1}{d})} \le \tau + O\Big(\frac{1}{d}\Big)\right], \end{aligned}$$

and, from (9),

$$1-r^2 = r^{\frac{2}{d+1}}(\lambda\kappa_d)^{-\frac{2}{d+1}}\left[1+O\left(\frac{\log d}{d}\right)\right],$$

with

$$\frac{1}{d}\log r = \frac{1}{d}\log\lambda\kappa_d + O\big(-\log(1-r^2)\big).$$

It easily follows that

$$\begin{cases} r^2 \sim \dfrac{2}{d} \log \lambda\kappa_d, & \text{if } \log \lambda\kappa_d \ll d, \\ r^2 \to 1 - \mathrm{e}^{-2x}, & \text{if } \log \lambda\kappa_d \sim dx \text{ with } x \in (0, \infty), \\ 1 - r^2 \sim (\lambda\kappa_d)^{-\frac{2}{d+1}}, & \text{if } \log \lambda\kappa_d \gg d, \end{cases}$$

which completes the proof. □

Theorem 3.2 and Lemma 3.1 allow us to complete the proof of Theorem 1.1 and of Theorem 1.2 in the case $m = 1$.

Proof of Theorem 1.1. It follows from Theorem 3.2 that

$$d\left(\log \frac{1}{\sqrt{1 - (h_\lambda^d)^2}} - \frac{1}{d+1} \log \lambda\kappa_d\right) + \log \sqrt{\mathfrak{m}(d)} = O_{\mathbb{P}}(1),$$

where $X(d) = O_{\mathbb{P}}(1)$ means that $\lim_{A\to\infty} \limsup_{d\to\infty} \mathbb{P}(|X(d)| > A) = 0$. Multiplying this equation by $-\frac{2}{d}$ and taking the exponential function, we deduce that

$$\left(1 - (h_\lambda^d)^2\right)(\lambda\kappa_d)^{\frac{2}{d+1}} = 1 + \frac{1}{\log d} \cdot O_{\mathbb{P}}(1),$$

where we discarded the $\mathfrak{m}(d)^{-\frac{1}{d}}$ term because $\mathfrak{m}(d) = O(d)$. In the subcritical regime $\log \lambda\kappa_d \ll d$, we obtain that in probability as $d \to \infty$,

$$h_\lambda^d \sim \sqrt{\frac{2}{d} \log \lambda\kappa_d}$$

(because then $1 - (\lambda\kappa_d)^{\frac{2}{d+1}} \sim -\frac{2}{d} \log \lambda\kappa d$). In the supercritical regime $\log \lambda\kappa_d \gg d$, we get instead

$$1 - h_\lambda^d \sim \frac{1}{2}(\lambda\kappa_d)^{-\frac{2}{d+1}}$$

(because $1 + h_\lambda^d \to 2$ in probability). This together with Lemma 3.1 completes the proof of Theorem 1.1. □

Proof of Theorem 1.2*, case* $m = 1$. Theorem 3.2 establishes Theorem 1.2 in the case $m = 1$, with

$$\mathfrak{a}(d; 1) := \log \sqrt{\mathfrak{m}(d)} - \frac{d}{d+1} \log \lambda\kappa_d, \tag{10}$$

and

$$\mathfrak{b}(d; 1) := -d. \tag{11}$$

□

4. The infimum of the support function over multiple directions

In this section, we extend the study of the asymptotic behavior of the support function when considered over several directions simultaneously. Namely, we consider

$$h_\lambda^{d,m} := \inf_{u \in \mathbb{S}^{d-1} \cap \mathbb{R}^m} h_\lambda^d(u),$$

where $m \geq 2$ is a fixed integer. Section 4.1 consists in reinterpreting the distribution of $h_\lambda^{d,m}$ in terms of a covering probability. We then present in Section 4.2 a remarkable covering technique due to Janson [18] which we specialize to our setting. This finally allows us to prove Theorem 1.2 in the case $m \geq 2$.

4.1. **Reduction to a covering problem.** We start by relating the tail event $h_\lambda^{d,m} \geq r$ to the event of covering the sphere $\mathbb{S}^{m-1} := \{y \in \mathbb{R}^m : \|y\| = 1\}$ with i.i.d. geodesic balls. In this direction, we write $\upsilon_{\mathbb{S}^{m-1}}(\mathrm{d}x)$ for the $(m-1)$-dimensional surface measure on $\mathbb{S}^{m-1}$ (so that $\upsilon_{\mathbb{S}^{m-1}}(\mathbb{S}^{m-1}) = m\kappa_m$), and

$$B_{\mathbb{S}^{m-1}}(x,\theta) := \Big\{ y \in \mathbb{R}^m : \|y\| = 1 \text{ and } \langle x, y\rangle > \cos\theta \Big\}$$

for the geodesic ball with center $x \in \mathbb{S}^{m-1}$ and radius $\theta \in (0,\pi]$ in $\mathbb{S}^{m-1}$.

Lemma 4.1 (Covering the sphere)**.** *For every $r \in (0,1)$, we have $h_\lambda^{d,m} \geq r$ if and only if*

$$\mathbb{S}^{m-1} = \bigcup_i B_{\mathbb{S}^{m-1}}(x_i, a\rho_i), \tag{12}$$

where

$$a := \frac{1}{\sqrt{d}} \frac{\sqrt{1-r^2}}{r}, \tag{13}$$

and the centers and radii (x_i, ρ_i) arise as the atoms of a Poisson point process on $\mathbb{S}^{m-1} \times (0,\infty)$ whose intensity $\Lambda_r^{d,m} \upsilon_{\mathbb{S}^{m-1}}(\mathrm{d}x) \otimes \mathbb{P}(R_r^{d,m} \in \mathrm{d}\rho)$ is given by

$$\Lambda_r^{d,m} := \lambda \kappa_d \cdot \frac{\mathrm{B}\big(1-r^2; 1+\frac{d-m}{2}, \frac{m}{2}\big)}{\mathrm{B}\big(1+\frac{d-m}{2}, \frac{m}{2}\big)}, \tag{14}$$

and, for every $\rho > 0$,

$$\mathbb{P}(R_r^{d,m} > \rho) := \frac{\mathrm{B}\big(1 - r^2 \cos^{-2}(a\rho); 1 + \frac{d-m}{2}, \frac{m}{2}\big)}{\mathrm{B}\big(1-r^2; 1+\frac{d-m}{2}, \frac{m}{2}\big)} \mathbb{1}_{\{\rho < \frac{\arccos r}{a}\}}. \tag{15}$$

Proof. Indeed, $h_\lambda^{d,m} \geq r$ if and only the sphere $r\mathbb{S}^{m-1}$ is entirely covered by the trace onto $\mathbb{R}^m$ of the Voronoi flower associated with K_λ^d, that is, by the spherical patches

$$\left(\frac{X'}{2} + \frac{\|X'\|}{2}\mathbb{B}^m\right) \cap r\mathbb{S}^{m-1} = rB_{\mathbb{S}^{m-1}}\left(\frac{X'}{\|X'\|}, R_{X'}\right),$$

where $R_{X'} := \arccos(\frac{r}{\|X'\|})$ and the points X' are the orthogonal projections of the points in $\mathcal{P}_\lambda^d \cap \mathcal{R}_r^{d,m}$, with

$$\mathcal{R}_r^{d,m} := \Big\{ x \in \mathbb{B}^d : x_1^2 + \cdots + x_m^2 \geq r^2 \Big\};$$

see Figure 2. Considering a point $X \in \mathcal{P}_\lambda^d \cap \mathcal{R}_r^{d,m}$, the norm of its orthogonal projection X' onto $\mathbb{R}^m$ has a law given by

$$\mathbb{E}\Big[g\Big(|X'|^2\Big)\Big] = \frac{\int_{r^2}^1 t^{\frac{m}{2}-1}(1-t)^{\frac{d-m}{2}} g(t)\,\mathrm{d}t}{\mathrm{B}\big(1-r^2; 1+\frac{d-m}{2}, \frac{m}{2}\big)}, \tag{16}$$

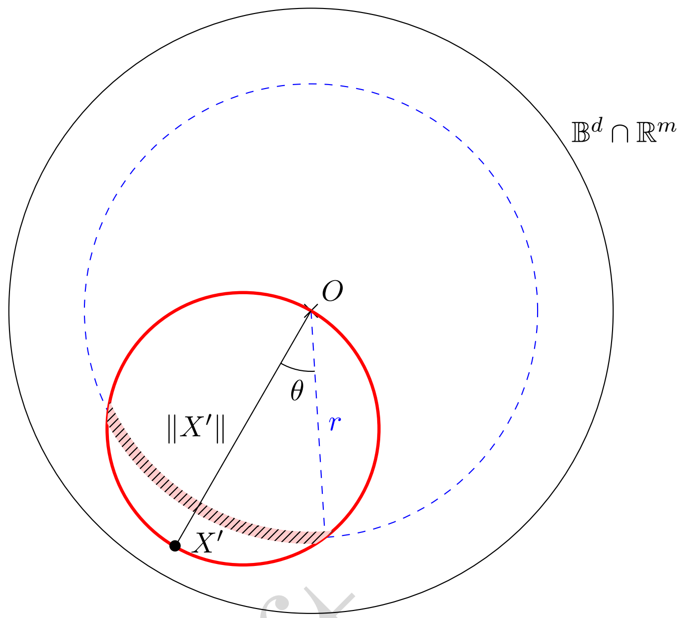

FIGURE 2. Projection onto $\mathbb{R}^m$. We have $h_\lambda^{d,m} \geq r$ if and only if the sphere $r\mathbb{S}^{m-1}$ (dashed) is covered by the union of its intersection with each petal (in red) of the Voronoi flower whose corresponding vertex lies in $\mathcal{R}_r^{d,m}$. Each projected vertex X' yields a geodesic ball (hatched) of radius $\theta := \arccos(r/|X'|)$.

for any measurable function $g\colon [0,\infty) \to [0,\infty)$. We further note that the projections X' are identically distributed, with $X'/\|X'\|$ uniform on $\mathbb{S}^{m-1}$ (by rotational invariance). Letting $R_r^{d,m}$ denote a random variable with law

$$R_r^{d,m} \overset{(d)}{=} \frac{1}{a} R_{X'} = \frac{1}{a} \arccos \frac{r}{\|X'\|}, \qquad \text{where} \quad a := \frac{1}{\sqrt{d}} \frac{\sqrt{1-r^2}}{r}, \tag{17}$$

we deduce that $h_\lambda^{d,m} \geq r$ if and only if $\mathbb{S}^{m-1}$ is covered by the geodesic balls $B_{\mathbb{S}^{m-1}}(x, a\rho)$, whose centers and radii (x,ρ) arise from a Poisson point process on $\mathbb{S}^{m-1} \times (0,\infty)$ with intensity

$$\lambda \cdot |\mathcal{R}_r^{d,m}| v_{\mathbb{S}^{m-1}}(\mathrm{d}x) \otimes \mathbb{P}(R_r^{d,m} \in \mathrm{d}\rho).$$

Now, the stated distribution function (15) of $R_r^{d,m}$ easily follows from (16) and (17):

$$\begin{aligned}
\mathbb{P}(R_r^{d,m} > \rho) &= \mathbb{P}\Big(\|X'\|^2 > r^2 \cos^{-2}(a\rho)\Big) \\
&= \frac{\int_{r^2}^1 t^{\frac{m}{2}-1}(1-t)^{\frac{d-m}{2}} \mathbb{1}_{\{t > r^2 \cos^{-2}(a\rho)\}} \,\mathrm{d}t}{\mathrm{B}\big(1-r^2; 1+\frac{d-m}{2}, \frac{m}{2}\big)} \\
&= \frac{\mathrm{B}\big(1-r^2\cos^{-2}(a\rho); 1+\frac{d-m}{2}, \frac{m}{2}\big)}{\mathrm{B}\big(1-r^2; 1+\frac{d-m}{2}, \frac{m}{2}\big)} \mathbb{1}_{\{\rho < \frac{\arccos r}{a}\}}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\lambda \cdot |\mathcal{R}_r^{d,m}| &= \lambda \int\!\!\cdots\!\!\int \mathbb{1}_{\{x_1^2+\cdots+x_m^2 \geq r^2\}}\, \kappa_{d-m} \big(1 - x_1^2 - \cdots - x_m^2\big)^{\frac{d-m}{2}} \mathrm{d}x_1 \cdots \mathrm{d}x_m \\
&= \frac{1}{2} \lambda m \kappa_m \kappa_{d-m} \int_{r^2}^1 t^{\frac{m}{2}-1} (1-t)^{\frac{d-m}{2}} \mathrm{d}t \\
&= \lambda \kappa_d \cdot \frac{\mathrm{B}\big(1-r^2; 1+\frac{d-m}{2}, \frac{m}{2}\big)}{\mathrm{B}\big(1+\frac{d-m}{2}, \frac{m}{2}\big)} \\
&=: \Lambda_r^{d,m},
\end{aligned}$$

where the second equality comes from the use of spherical coordinates, and the third equality is due to the expression (6) for the volume of Euclidean balls and to the relation $\mathrm{B}(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ between the beta and gamma functions. $\square$

We are therefore reduced to understanding the probability as $d \to \infty$ of the covering event (12). As we will see in the next section, it turns out that Janson [18] derived precise estimates for the probability of covering a manifold of fixed dimension using a Poisson process of i.i.d. patches whose intensity Λ increases to infinity as the scale parameter a decreases to 0. We conclude this section by showing that the random radii $R_r^{d,m}$, $r \in (0,1)$, have a common limit distribution as $d \to \infty$. This convergence will hold with respect to the Wasserstein metric W_1, which for two real random variables X and Y is given by

$$W_1(X,Y) := \int_0^\infty \big|\mathbb{P}(X > t) - \mathbb{P}(Y > t)\big| \,\mathrm{d}t.$$

We recall that the W_1-convergence amounts to the convergence in distribution together with the convergence of the first moment.

Lemma 4.2 (Convergence of the patch radii)**.** *Let $m \geq 2$ be a fixed integer, let $r := r(d) \in (0,1)$ and let $a := \sqrt{1-r^2}/r\sqrt{d}$. Suppose that*

$$d \gg \frac{1}{r^2} + \log^2 \frac{1}{1-r}, \quad \textit{or equivalently,}\ a + \frac{1}{d} \log^2 \frac{1}{a} \to 0. \tag{18}$$

Then

$$W_1(R_r^{d,m}, R) = O\bigg(\frac{1}{dr^2} \log(dr^2)\bigg), \tag{19}$$

where R is a standard Rayleigh random variable, with Lebesgue density $\rho \mapsto \rho\, \mathrm{e}^{-\rho^2/2}$ on $(0,\infty)$ and moments

$$\mathbb{E}\, R^k = 2^{\frac{k}{2}}\, \Gamma\bigg(1 + \frac{k}{2}\bigg), \qquad k \in \mathbb{Z}_+. \tag{20}$$

Moreover, the convergence holds with the following upper bound: for every $w > 0$ and all d sufficiently large,

$$\sup_{\rho > 0} \frac{\mathbb{P}(R_r^{d,m} > \rho)}{\mathbb{P}\Big(\Big[1 + \frac{w}{\log \frac{1}{a}}\Big] R > \rho\Big)} \leq 1. \tag{21}$$

Proof. We observe from (15) that

$$\mathbb{P}(R_r^{d,m} > \rho) = \frac{\mathrm{B}\big(1 - r^2 \cos^{-2}(a\rho); 1 + \frac{d-m}{2}, \frac{m}{2}\big)}{\mathrm{B}(1 - r^2; 1 + \frac{d-m}{2}, \frac{m}{2})} \mathbb{1}_{\{\rho < \frac{\arccos r}{a}\}}$$

is a positive and continuously differentiable of $x := \tan^{-2}(a\rho) = \cos^{-2}(a\rho) - 1$ on the domain $(0, r^{-2} - 1)$. Letting $F(x) := \log \mathrm{B}(1 - r^2 - r^2\, x; 1 + \frac{d-m}{2}, \frac{m}{2})$ allows us to write $\log \mathbb{P}(R_r^{d,m} > \rho) = F(x) - F(0)$, so that by the mean value theorem there exists $\bar{x} \in (0, x)$ with

$$\begin{aligned}
\log \mathbb{P}(R_r^{d,m} > \rho) &= x F'(\bar{x}) \\
&= -r^2\, x\, \frac{\big(1 - r^2 - r^2 \bar{x}\big)^{\frac{d-m}{2}} r^{m-2}\, \bar{x}^{\frac{m}{2}-1}}{\mathrm{B}(1 - r^2 - r^2\, \bar{x}; 1 + \frac{d-m}{2}, \frac{m}{2})}.
\end{aligned}$$

Applying Lemma 2.2 for the denominator, we get (introducing $r^{-2} - 1 = da^2$)

$$\begin{aligned}
\log \mathbb{P}(R_r^{d,m} > \rho) &= -\left(1 + \frac{d-m}{2}\right) \frac{r^2\, x}{1 - r^2 - r^2\, \bar{x}} \left[1 + O\left(\frac{1 - r^2 - r^2\, \bar{x}}{dr^2(1 + \bar{x})}\right)\right] \\
&= -\frac{x}{2a^2}\left(1 - \frac{\bar{x}}{da^2}\right)^{-1} \left[1 + O\left(\frac{1}{dr^2}\right)\right],
\end{aligned}$$

where the error term is uniform in $x \in (0, r^{-1} - 1]$. Recalling $\bar{x} < x = \tan^2(a\rho)$ and using the inequality $y < \tan y < y(1 - \frac{4}{\pi^2} y^2)^{-1}$ (see, e.g., [3]) with $y := a\rho \in (0, \arccos r) \subset (0, \frac{\pi}{2})$, we then obtain

$$-\frac{\rho^2}{2}\left[\left(1 - \frac{4a^2\rho^2}{\pi^2}\right)^2 - \frac{\rho^2}{d}\right]^{-1} \left[1 + O\left(\frac{1}{dr^2}\right)\right] \le \log \mathbb{P}(R_r^{d,m} > \rho) \le -\frac{\rho^2}{2}\left[1 + O\left(\frac{1}{dr^2}\right)\right], \tag{22}$$

uniformly in $0 \le \rho < A$, where

$$A := \frac{\arccos r}{a} = r\sqrt{d}\, \frac{\arccos r}{\sqrt{1 - r^2}} \asymp r\sqrt{d}$$

tends to infinity by (18). Observing that the tail of the Rayleigh variable R fulfills $\log \mathbb{P}(R > \rho) = -\frac{\rho^2}{2}$, it easily follows that, for $B := \log(dr^2)$,

$$\frac{\mathbb{P}(R_r^{d,m} > \rho)}{\mathbb{P}(R > \rho)} \mathbb{1}_{\{0 \le \rho < \sqrt{B}\}} = 1 + O\big(B^2\, \mathrm{e}^{-B}\big),$$

and

$$\frac{\mathbb{P}(R_r^{d,m} > \rho)}{\mathbb{P}(R > \rho)} \mathbb{1}_{\{\sqrt{B} \le \rho < A\}} = O(1),$$

where the error terms do not depend on ρ. We deduce that

$$
\begin{aligned}
W_1(R_r^{d,m}, R) &= \int_0^\infty \Big|\mathbb{P}(R_r^{d,m} > \rho) - \mathbb{P}(R > \rho)\Big| \,\mathrm{d}\rho \\
&\le \int_0^{\sqrt{B}} \Big|\mathbb{P}(R_r^{d,m} > \rho) - \mathbb{P}(R > \rho)\Big| \,\mathrm{d}\rho + \int_{\sqrt{B}}^\infty \Big(\mathbb{P}(R_r^{d,m} > \rho) + \mathbb{P}(R > \rho)\Big)\mathrm{d}\rho \\
&= O\big(B^2 \,\mathrm{e}^{-B}\big) + O(1) \int_{\sqrt{B}}^\infty \mathrm{e}^{-\frac{\rho^2}{2}} \,\mathrm{d}\rho \\
&= O\bigg(\frac{1}{dr^2} \log^2(dr^2)\bigg),
\end{aligned}
$$

as stated. Finally, since

$$
\bigg(1 + \frac{w}{\log \frac{1}{a}}\bigg)^{-\frac{1}{2}} \;=\; 1 + O(a^2) \;=\; 1 + O\bigg(\frac{1}{dr^2}\bigg)
$$

for every fixed $w > 0$, we also obtain from the upper bound in (22) that (21) holds. For the moments of the standard Rayleigh distribution, see, e.g., [19, § 18.3]. □

4.2. **Application of Janson's covering result.** In [18], Janson showed that the renormalized number of random "small sets" needed to cover a fixed "big set" converges, when properly normalized, to the Gumbel extreme value distribution as the "size" of the small sets tends to zero. More precisely, let the big set be a $\mathcal{C}^2$, D-dimensional compact Riemannian manifold M with volume measure v_M, and suppose that the small sets are i.i.d. geodesic balls, that is, they are all of the form $B_M(x_i, a\rho_i)$, $i \ge 1$, where $a > 0$ is a vanishing scale parameter, and the centers and radii (x_i, ρ_i) arise as the atoms of a Poisson point process on $M \times (0,\infty)$ with intensity $\Lambda v_M(\mathrm{d}x) \otimes \mathbb{P}(R \in \mathrm{d}\rho)$, for some positive random variable R and $\Lambda := \Lambda(a) \to \infty$ as $a \to 0$. Then, denoting by $\mathrm{Cover}(\Lambda, R, a; M)$ the event

$$
M = \bigcup_{i \ge 1} B_M(x_i, a\rho_i),
$$

Janson [18, Lemma 8.1] proved that

$$
\lim_{a\to 0} \mathbb{P}\big(\mathrm{Cover}(\Lambda, R, a; M)\big) = \mathrm{e}^{-\,\mathrm{e}^{-\tau}}, \tag{J}
$$

under the following two conditions:

$$
\mathbb{E}\, R^q < \infty \quad \text{for some } q > D, \tag{J_1}
$$

and

$$
J(\Lambda, R, a; M) := ba^D v_M(M)\,\Lambda - \log\frac{1}{ba^D} - D\log\log\frac{1}{ba^D} - \log\alpha(R) \xrightarrow[a\to 0]{} \tau \in \mathbb{R}, \tag{J_2}
$$

with

$$
b := b(R; M) := \frac{\pi^{D/2}\,\mathbb{E}\, R^D}{\Gamma(1 + \frac{D}{2}) v_M(M)}, \tag{J_b}
$$

and [18, Eq. (9.24)]

$$\alpha(R) := \frac{1}{D!}\left(\frac{\sqrt{\pi}\,\Gamma(1+\frac{D}{2})}{\Gamma(\frac{D+1}{2})}\right)^{D-1}\frac{\left(\mathbb{E}\,R^{D-1}\right)^{D}}{\left(\mathbb{E}\,R^{D}\right)^{D-1}}. \tag{J_α}$$

In view of Lemma 4.1, we would like to estimate the probability of the event $\mathrm{Cover}(\Lambda_r^{d,m}, R_r^{d,m}, a; \mathbb{S}^{m-1})$, where the scale parameter $a := \sqrt{1-r^2}/r\sqrt{d}$ vanishes as $d \to \infty$. Although Janson's original result is stated only when the distribution of the random radius does not depend on a, we show in Lemma 4.3 below that it still holds when we use the radii $R_r^{d,m}$ instead of their Rayleigh limit R in Lemma 4.2. This is done thanks to a slight improvement of Janson's result, Proposition A, which may be of independent interest (see Appendix).

Lemma 4.3 (Application of the extension of Janson's result)**.** *With the notation of Lemmas 4.1 and 4.2, suppose (J_2) holds with $\Lambda := \Lambda_r^{d,m}$ and $M := \mathbb{S}^{m-1}$ (i.e., $J\big(\Lambda_r^{d,m}, R, a; \mathbb{S}^{m-1}\big) \to \tau$), and suppose also that (18) holds. Then*

$$\lim_{d\to\infty} \mathbb{P}\big(\mathrm{Cover}(\Lambda_r^{d,m}, R_r^{d,m}, a; \mathbb{S}^{m-1})\big) = \mathrm{e}^{-\mathrm{e}^{-\tau}}.$$

Proof. We apply Proposition A in Appendix. We have (J_2) for $\Lambda := \Lambda^{d,m}$ and $D := m-1$, and also (J_1) because all moments of the Rayleigh distribution are finite. It remains to show that the two conditions of Proposition A related to $R_a := R_r^{d,m}$ are satisfied. The first one is (21) given by Lemma 4.2. For the second one, we can take $\varepsilon_a := \log^{-\frac{3}{4}}(\frac{1}{a})$. Indeed, we have,

$$\mathbb{P}(R \le \varepsilon_a) = 1 - \exp\left(-\frac{\varepsilon_a^2}{2}\right) = O(\varepsilon_a^2) = o\left(\frac{1}{\log\frac{1}{a}}\right).$$

Further, (19) gives

$$W_1(R_r^{d,m}, R) = O\left(\frac{1}{dr^2}\log^2(dr^2)\right) = o\left(\frac{\varepsilon_a}{\log\frac{1}{a}}\right),$$

since, by (18),

$$\frac{1}{dr^2}\log^2(dr^2)\,\frac{\log\frac{1}{a}}{\varepsilon_a} = \left[\left(\frac{1}{dr^2}\right)\log^2(dr^2)\right]^{\frac{1}{8}} \cdot \left[\frac{1}{d}\log^2\frac{1}{a} + a^2\log^2\frac{1}{a}\right]^{\frac{7}{8}} \to 0. \qquad \square$$

The applicability of Lemma 4.3 is done in Lemma 4.4 below. Plugging in the expressions of $\mathbb{E}\,R^k$ in (20) into (J_α), we can see that $\alpha(R)$ simplifies to

$$\alpha := \alpha(R) = \frac{\pi^{\frac{m-1}{2}}\,\Gamma(\frac{m+1}{2})}{(m-1)!} = \frac{\pi^{\frac{m}{2}}}{2^{m-1}\,\Gamma(\frac{m}{2})}, \tag{23}$$

by an application of Legendre's duplication formula. Similarly, using also the expression of κ_m in (6) and the fundamental property $\Gamma(1+\frac{m}{2}) = \frac{m}{2}\Gamma(\frac{m}{2})$, the expression of $b(R;M)$ in (J_b) reduces to

$$b := b(R;\mathbb{S}^{m-1}) = \frac{\left(\sqrt{2\pi}\right)^{m-1}\Gamma(\frac{m}{2})}{2\pi^{\frac{m}{2}}} = \frac{2^{\frac{m-3}{2}}\,\Gamma(\frac{m}{2})}{\sqrt{\pi}}. \tag{24}$$

Lemma 4.4 (Verification of Janson's condition (J_2) and of (18))**.** *Let $m \geq 2$ be a fixed integer, and suppose that one of the three assumptions* ($\mathrm{A}_{\mathrm{sub}}$), ($\mathrm{A}_{\mathrm{crit}}$), *or* ($\mathrm{A}_{\mathrm{sup}}$) *below occurs:*

$$\log^2 d \ll \log \lambda\kappa_d \ll d, \tag{$\mathrm{A}_{\mathrm{sub}}$}$$

$$\log \lambda\kappa_d \sim dx \text{ with } x \in (0,\infty), \tag{$\mathrm{A}_{\mathrm{crit}}$}$$

$$1 \ll \frac{1}{d}\log \lambda\kappa_d - \frac{1}{2}\log\log d \ll \sqrt{d}. \tag{$\mathrm{A}_{\mathrm{sup}}$}$$

Let α and b as in (23) and (24). Then for every $\tau \in \mathbb{R}$, there exists $r := r(d;\tau) > 0$ such that, for $a := a(d)$ and $\Lambda_r^{d,m}$ as in (13) and (14), condition (J_2) holds:

$$J(\Lambda_r^{d,m}, R, a; \mathbb{S}^{m-1}) = ba^{m-1}\, m\kappa_m \Lambda_r^{d,m} + \log\bigl(ba^{m-1}\bigr) - (m-1)\log\bigl[-\log\bigl(ba^{m-1}\bigr)\bigr] - \log\alpha \to \tau. \tag{25}$$

Furthermore, (18) holds: $a + \frac{1}{d}\log^2\frac{1}{a} \to 0$.

Proof. According to (13), we seek $r := (1+da^2)^{-\frac{1}{2}} \in (0,1)$. We start with the following asymptotics of (14),

$$\begin{aligned}
\Lambda_r^{d,m} &= \lambda\kappa_d \,\frac{(1-r^2)^{1+\frac{d-m}{2}}\, r^{m-2} (\frac{d}{2})^{\frac{m}{2}-1}}{\Gamma(\frac{m}{2})}\left[1 + O\left(\frac{1}{dr^2}\right)\right] \\
&= \lambda\kappa_d\, d^{\frac{d}{2}}\,\frac{a^{d+2-m}(1+da^2)^{-\frac{d}{2}}}{2^{\frac{m}{2}-1}\,\Gamma(\frac{m}{2})}\left[1 + O\left(\frac{1}{d} + a^2\right)\right],
\end{aligned}$$

obtained by Lemma 2.2, provided that $\frac{1}{dr^2} = \frac{1}{d} + a^2 \ll 1$. In this case,

$$ba^{m-1}\, m\kappa_m\, \Lambda_r^{d,m} = \frac{\sqrt{2}\,\pi^{\frac{m-1}{2}}}{\Gamma(\frac{m}{2})}\,\lambda\kappa_d\,\frac{a}{(1+\frac{1}{da^2})^{d/2}}\left[1 + O\left(\frac{1}{d} + a^2\right)\right], \tag{26}$$

Next,

$$\log\bigl(ba^{m-1}\bigr) = -(m-1)\log\frac{1}{a} + \log b, \tag{27}$$

and

$$-(m-1)\log\bigl[-\log\bigl(ba^{m-1}\bigr)\bigr] = -(m-1)\log\log\frac{1}{a} - (m-1)\log(m-1) + o(1). \tag{28}$$

Adding (26), (27), (28) and $-\log\alpha$ yields

$$ba^{m-1}\, m\kappa_m \Lambda_r^{d,m} + \log\bigl(ba^{m-1}\bigr) - (m-1)\log\bigl[-\log\bigl(ba^{m-1}\bigr)\bigr] - \log\alpha \tag{29}$$

$$= \frac{\sqrt{2}\,\pi^{\frac{m-1}{2}}}{\Gamma(\frac{m}{2})}\,\lambda\kappa_d\,\frac{a}{(1+\frac{1}{da^2})^{d/2}}\left[1 + O\left(\frac{1}{d} + a^2\right)\right] - (m-1)\log\frac{1}{a} - (m-1)\log\log\frac{1}{a} - \log B_m + o(1), \tag{30}$$

where

$$\frac{\alpha\,(m-1)^{m-1}}{b} = \frac{\pi^{\frac{m+1}{2}}(m-1)^{m-1}}{2^{\frac{3m-5}{2}}\Gamma(\frac{m}{2})^2} =: B_m \tag{31}$$

follows from the expressions of α in (23) and b in (24). We now observe by monotonicity and continuity in the variable $a \in (0,1)$ that the equation

$$\frac{\sqrt{2}\,\pi^{\frac{m-1}{2}}}{\Gamma(\frac{m}{2})}\,\lambda\kappa_d\,\frac{a}{\left(1+\frac{1}{da^2}\right)^{d/2}} = (m-1)\log\frac{1}{a} + (m-1)\log\log\frac{1}{a} + \log B_m + \tau \tag{32}$$

has an asymptotically unique solution $a := a(d;\tau)$, which tends to 0 because $\lambda\kappa_d \to \infty$. Passing to the logarithm easily yields

$$\frac{d}{2}\log\left(1+\frac{1}{da^2}\right) = \log\frac{A_m\,\lambda\kappa_d}{\frac{1}{a}\log\frac{1}{a}} - \frac{(m-1)\log\log\frac{1}{a} + \log B_m + \tau}{(m-1)\log\frac{1}{a}} + o\left(\frac{1}{\log\frac{1}{a}}\right)$$

with

$$A_m := \frac{\sqrt{2}\,\pi^{\frac{m-1}{2}}}{(m-1)\Gamma(\frac{m}{2})}. \tag{33}$$

Dropping the $O(1)$ terms, we have in particular

$$\frac{1}{d}\log\frac{1}{a} + \frac{1}{d}\log\log\frac{1}{a} + O\left(\frac{1}{d}\right) = \frac{1}{d}\log\lambda\kappa_d - \frac{1}{2}\log\left(1+\frac{1}{da^2}\right). \tag{34}$$

If $\log\lambda\kappa_d \ll d$ holds, then $\frac{1}{d}\log\frac{1}{a} \to 0$, and then $\log(1+\frac{1}{da^2}) \to 0$. This shows that $da^2 \to \infty$ and

$$\frac{1}{d}\log\lambda\kappa_d - \frac{1}{2da^2}(1+o(1)) = O\left(\frac{\log d}{d}\right).$$

With $\log^2 d \ll \log\lambda\kappa_d$, this implies $\frac{1}{a} = O(\sqrt{\log\lambda\kappa_d})$, then iterating (34) yields $\frac{1}{a} \sim \sqrt{2\log\lambda\kappa_d}$, and

$$\log\frac{1}{a} = \log\sqrt{d\left((\lambda\kappa_d)^{\frac{2}{d}}-1\right)} + O\left(\frac{\log\log\lambda\kappa_d}{\log\lambda\kappa_d}\right) \qquad \text{(subcritical regime)}.$$

Similarly, if $\frac{1}{d}\log\lambda\kappa_d \to x \in (0,\infty)$, then (34) implies that $\log\frac{1}{a} = O(d)$, which in turn implies that $\log\left(1+\frac{1}{da^2}\right) = O(1)$, meaning that $a \gtrsim \frac{1}{\sqrt{d}}$. This forces $\frac{1}{d}\log\lambda\kappa_d - \frac{1}{2}\log\left(1+\frac{1}{da^2}\right) \to 0$, or in other words,

$$\frac{1}{a} \sim \sqrt{d\left((\lambda\kappa_d)^{\frac{2}{d}}-1\right)} \sim \sqrt{d(\mathrm{e}^{2x}-1)}.$$

One more iteration of (34) gives

$$\log\frac{1}{a} = \log\sqrt{d(\mathrm{e}^{2x}-1)} + O\left(\frac{\log d}{d}\right) \qquad \text{(critical regime)}.$$

Lastly, if $d \ll \log\lambda\kappa_d \ll d^{\frac{3}{2}}$ holds, then (34) forces $da^2 \to 0$ and

$$\left(1+\frac{1}{d}\right)\log\frac{1}{a} + O\left(\frac{\log\log\frac{1}{a}}{d}\right) = \log\left((\lambda\kappa_d)^{\frac{1}{d}}\sqrt{d}\right) - \frac{1}{2}\log(1+da^2),$$

which implies $\frac{1}{a} \sim (\lambda\kappa_d)^{\frac{1}{d}}\sqrt{d}$ and also $\frac{1}{d}\log^2\frac{1}{a} \to 0$. Plugging this back into the previous equation, we find

$$\begin{aligned}\log\frac{1}{a} &= \left(1-\frac{1}{d}\right)\log\left((\lambda\kappa)^{\frac{1}{d}}\sqrt{d}\right) + O\left(\frac{\log d}{d} \vee (\lambda\kappa_d)^{-\frac{2}{d}}\right)\\ &= \log\sqrt{d\left((\lambda\kappa_d)^{\frac{2}{d}}-1\right)} + o\left(\frac{1}{\log\frac{1}{a}}\right) \qquad \text{(supercritical regime)},\end{aligned}$$

where the second asymptotic equality holds under $1 \ll \frac{1}{d}\log\lambda\kappa_d - \frac{1}{2}\log\log d \ll \sqrt{d}$. In particular, $\frac{1}{d}\log^2\frac{1}{a} \ll 1$ holds. Finally, if we plug in (32) into (30), we obtain that (29) reduces to

$$ba^{m-1}\, m\kappa_m\Lambda_r^{d,m} + \log\bigl(ba^{m-1}\bigr) - (m-1)\log\bigl[-\log\bigl(ba^{m-1}\bigr)\bigr] = \tau + O\biggl(\log\frac{1}{a}\cdot\biggl[\frac{1}{d}+a^2\biggr]\biggr),$$

where the error term is a $o(1)$ because (18) holds under each of the three stated conditions ($\mathrm{A}_{\mathrm{sub}}$), ($\mathrm{A}_{\mathrm{crit}}$) and ($\mathrm{A}_{\mathrm{sup}}$). □

Scholium 4.5 (Summary of useful asymptotics derived in the proof of Lemma 4.4)**.** *The sequence a fulfills the implicit asymptotic equality*

$$\frac{d}{2}\log\biggl(1+\frac{1}{da^2}\biggr) = \log\frac{A_m\,\lambda\kappa_d}{\frac{1}{a}\log\frac{1}{a}} - \frac{(m-1)\log\log\frac{1}{a}+\log B_m+\tau}{(m-1)\log\frac{1}{a}} + o\biggl(\frac{1}{\log\frac{1}{a}}\biggr), \tag{35}$$

where we record

$$A_m := \frac{\sqrt{2}\,\pi^{\frac{m-1}{2}}}{(m-1)\Gamma(\frac{m}{2})}, \qquad B_m := \frac{\pi^{\frac{m+1}{2}}(m-1)^{m-1}}{2^{\frac{3m-5}{2}}\Gamma(\frac{m}{2})^2}, \tag{36}$$

from (33) *and* (31)*. Furthermore we have, in all regimes,*

$$\log\frac{1}{a} = \mathfrak{s}(d) + o\biggl(\frac{1}{\mathfrak{s}(d)}\biggr), \quad \textit{where } \mathfrak{s}(d) := \log\sqrt{d\Bigl((\lambda\kappa_d)^{\frac{2}{d}}-1\Bigr)}. \tag{37}$$

We are now ready to prove Theorem 1.2, in the case $m \ge 2$.

Proof of Theorem 1.2*, case $m \ge 2$.* For $\tau\in\mathbb{R}$, $a := a(d;\tau)$, and $r := (1+da^2)^{-\frac{1}{2}}$ as in Lemma 4.4, the conditions to apply Lemma 4.3 are in place, hence (recalling (12))

$$\lim_{d\to\infty}\mathbb{P}(h_\lambda^{d,m}\ge r) = \mathrm{e}^{-\mathrm{e}^{-\tau}}.$$

Now, notice that

$$\begin{aligned}\mathbb{P}(h_\lambda^{d,m}\ge r) &= \mathbb{P}\Bigl(h_\lambda^{d,m}\ge(1+da^2)^{-\frac{1}{2}}\Bigr)\\ &= \mathbb{P}\biggl[-d\log\Bigl(1-(h_\lambda^{d,m})^2\Bigr)\ge\frac{d}{2}\log\biggl(1+\frac{1}{da^2}\biggr)\biggr].\end{aligned}$$

Plugging in (35), we find

$$\mathbb{P}(h_\lambda^{d,m}\ge r) = \mathbb{P}\Biggl[\log\frac{A_m\,\lambda\kappa_d}{\frac{1}{a}\log\frac{1}{a}} - d\log\frac{1}{\sqrt{1-(h_\lambda^{d,m})^2}} \le \frac{(m-1)\log\log\frac{1}{a}+\log B_m+\tau}{(m-1)\log\frac{1}{a}} + o\biggl(\frac{1}{\log\frac{1}{a}}\biggr)\Biggr].$$

To conclude, it remains to express $\log\frac{1}{a}$ and $\log\log\frac{1}{a}$ in terms of $\mathfrak{s}(d) := \log\mathfrak{r}(d)$. According to (37),

$$\log\frac{1}{a} = \mathfrak{s}(d) + o\biggl(\frac{1}{\mathfrak{s}(d)}\biggr),$$

and so

$$\log\log\frac{1}{a} = \log\mathfrak{s}(d) + o\biggl(\frac{1}{\mathfrak{s}(d)}\biggr).$$

Hence

$$(m-1)d\,\mathfrak{s}(d)\Biggl[\frac{1}{d}\log\frac{A_m\,\lambda\kappa_d}{\mathfrak{r}(d)\mathfrak{s}(d)} - \log\frac{1}{\sqrt{1-(h_\lambda^{d,m})^2}}\Biggr] - (m-1)\log\mathfrak{s}(d) - \log B_m$$

converges in law to the standard Gumbel distribution. This establishes Theorem 1.2 with

$$\mathfrak{a}(d;m) := (m-1)\mathfrak{s}(d)\log\frac{A_m\,\lambda\kappa_d}{\mathfrak{r}(d)\mathfrak{s}(d)} - (m-1)\log\mathfrak{s}(d) - \log B_m, \tag{38}$$

and

$$\mathfrak{b}(d;m) := (m-1)d\,\mathfrak{s}(d). \tag{39}$$

□

5. Consequences on the radius-vector function

To conclude this work, we prove an easy consequence on the asymptotics of the radius-vector function ρ_λ^d given at (2). Recall that $\rho_\lambda^d \le h_\lambda^d$, see Figure 1.

Corollary 5.1 (Subcritical and critical regimes for ρ_λ^d)**.** *Let Assumption (H) hold.*

(i) In the subcritical regime, under the condition ($\mathrm{A}_{\mathrm{sub}}$),

$$\limsup_{d\to\infty}\sqrt{\frac{d}{2\log\lambda\kappa_d}}\,\rho_\lambda^d \le 1, \quad \textit{in probability.}$$

(ii) In the supercritical regime, under the condition ($\mathrm{A}_{\mathrm{sup}}$),

$$\lim_{d\to\infty}(\lambda\kappa_d)^{\frac{2}{d+1}}\Big(1-\rho_\lambda^d\Big) = \frac{1}{2}, \quad \textit{in probability.}$$

Proof. Since $\rho_\lambda^d \le h_\lambda^d$, (i) is an immediate consequence of Theorem 1.1, and (ii) will follow from Theorem 3.2 if we prove that, for every $\varepsilon \in (0,1)$,

$$\mathbb{P}\bigg((\lambda\kappa_d)^{\frac{2}{d+1}}\Big(1-\rho_\lambda^d(u)\Big) > \frac{1}{2}+\varepsilon\bigg) \xrightarrow[d\to\infty]{} 0.$$

Let $\{X_1,\dots,X_N\} = \mathcal{P}_\lambda^d \cap \mathcal{C}^d(r;u)$ and denote by X_i', $1 \le i \le n$, their projections onto the $(d-1)$-dimensional hyperplane $\{x \in \mathbb{R}^d : \langle x,u\rangle = r\}$, with $r \in (0,1)$ arbitrary. The number N has a Poisson distribution with parameter $\ell(r) := \lambda|\mathcal{C}^d(r;u)|$, and conditional on N, the points $X_i' - ru$ are i.i.d. according to a symmetric distribution on $\mathbb{R}^{d-1}$. First, Wendel's formula [30] allows us to write

$$\begin{aligned}\mathbb{P}\Big(0 \notin \operatorname{conv}\{X_1',\dots,X_N'\}\;\Big|\;N\Big) &= \mathbb{1}_{\{N<d\}} + \mathbb{1}_{\{N\ge d\}}\,2^{-(N-1)}\sum_{k=0}^{d-2}\binom{N-1}{k}\\ &= 1 - \mathbb{1}_{\{N\ge d\}}\,2^{-(N-1)}\sum_{k=d}^{N}\binom{N-1}{k-1}.\end{aligned}$$

Next, we observe that $ru \notin \operatorname{conv}\{X_1',\dots,X_N'\}$ on the event $\{\rho_\lambda^d(u) \le r\}$, so we get

$$\begin{aligned}\mathbb{P}\Big(\rho_\lambda^d(u) \le r\Big) &\le \mathbb{E}\Big[\mathbb{P}\Big(0 \notin \operatorname{conv}\{X_1',\dots,X_N'\}\;\Big|\;N\Big)\Big]\\ &= 1 - \mathbb{E}\bigg[\mathbb{1}_{\{N\ge d\}}\,2^{-(N-1)}\sum_{k=d}^{N}\binom{N-1}{k-1}\bigg].\end{aligned} \tag{40}$$

We now let r depend on d and observe that, for $r := 1 - (\lambda\kappa_d)^{-\frac{2}{d+1}}(\frac{1}{2}+\varepsilon)$, we have

$$\rho_\lambda^d(u) \le r \iff (\lambda\kappa_d)^{\frac{2}{d+1}}\Big(1-\rho_\lambda^d(u)\Big) > \frac{1}{2}+\varepsilon,$$

so it suffices to prove that the upper bound in (40) goes to 0, where N is a Poisson r.v. with mean $\ell(r)$. But

$$\begin{aligned}\ell(r) &= \frac{\lambda\kappa_{d-1}}{2}\,\mathrm{B}\big(1-r^2; \tfrac{d+1}{2}, \tfrac{1}{2}\big)\\ &= -\log \mathbb{P}\Big(h_\lambda^d \le r\Big)\\ &= \exp\Big\{\log\lambda\kappa_d + \frac{d+1}{2}\log(1-r^2) - \log r\sqrt{2\pi d} + o(1)\Big\},\end{aligned}$$

by (3), (4), and (8), provided that $d \gg r^{-2}$. In fact,

$$1-r^2 = (\lambda\kappa_d)^{-\frac{2}{d+1}}(1+2\varepsilon)\Big[1+O\Big((\lambda\kappa_d)^{-\frac{2}{d+1}}\Big)\Big],$$

so

$$\frac{d+1}{2}\log(1-r^2) = -\log\lambda\kappa_d + \frac{d+1}{2}\log(1+2\varepsilon) + o(1),$$

because $\frac{1}{d}\log\lambda\kappa_d - \frac{1}{2}\log d \gg 1$, which by the way shows that $-d\log(1-r^2)\to\infty$, so $dr^2\to\infty$, too. We can then see that $\ell(r) = (1+2\varepsilon)^{\frac{d}{2}+o(d)} \gg d$ and so, as in the proof of Lemma 2.1, $N \gg d$, w.h.p. Hence

$$\sum_{k=d}^{N}\binom{N-1}{k-1} \sim \sum_{k=0}^{N-1}\binom{N-1}{k} = 2^{N-1},$$

and we conclude from the dominated convergence theorem that (40) vanishes. $\square$

Appendix

In this appendix, we state and prove Proposition A, which is instrumental in deriving Theorem 1.2. As exposed in Section 4.2, Janson's result [18, Lemma 8.1] says that, under (J_1) and (J_2), the probability $\mathbb{P}\big(\mathrm{Cover}(\Lambda(a), R, a; M)\big)$ of covering the manifold M by a union of geodesic balls, $\bigcup_i B_M(x_i, a\rho_i) = M$, where the (x_i,ρ_i)'s arise as the atoms of a Poisson point process with intensity $\Lambda(a)v_M(\mathrm{d}x)\otimes\mathbb{P}(R\in\mathrm{d}\rho)$, converges to the Gumbel distribution function as $a\to 0$. In this section, we prove the following extension where the random radius R is allowed to depend on a. Instead of adapting Janson's result and rewriting the whole proof, we rather reduce to it using a coupling argument. Namely, if R_a converges to R in such a way that we may construct (R, R_a) so that

$$(1-\eta)R \le R_a \le (1+\eta)R \tag{41}$$

holds with high probability for some positive sequence $\eta := \eta(a)$ vanishing sufficiently fast, then we may approximate (from above and below) $\mathbb{P}\big(\mathrm{Cover}(\Lambda, R_a, a; M)\big)$ with similar probabilities where R_a is changed to $(1\pm\eta)R$, which after replacing a by $a/(1\pm\eta)$ leads to the case handled by Janson.

In Proposition A below, condition (42) allows us to realize the upper bound in (41) and is of course satisfied if the stronger condition ($\forall\rho>0,\ \mathbb{P}(R_a>\rho)\le\mathbb{P}(R>\rho)$) holds for $a>0$ sufficiently small (e.g., if the sequence of functions $\rho \mapsto \mathbb{P}(R_a > \rho)$ is eventually non-decreasing as $a \to 0$), while condition (43) allows us to realize the lower bound in (41).

Proposition A (Extension of Janson's result)**.** *Let M be a $\mathcal{C}^2$, D-dimensional compact Riemannian manifold, and let $\Lambda := \Lambda(a) > 0$ and R fulfill Janson's conditions (J_1) and (J_2). Suppose R_a, $a > 0$, are positive random variables such that for every $w > 0$ and all a sufficiently small,*

$$\sup_{\rho>0} \frac{\mathbb{P}(R_a > \rho)}{\mathbb{P}\Big(\Big[1 + \frac{w}{\log \frac{1}{a}}\Big] R > \rho\Big)} \leq 1. \tag{42}$$

Suppose further that there exists $\varepsilon := \varepsilon(a) > 0$ such that

$$\mathbb{P}(R \leq \varepsilon_a) = o\left(\frac{1}{\log \frac{1}{a}}\right), \quad \textit{and} \quad W_1(R_a, R) = O\left(\frac{\varepsilon_a}{\log \frac{1}{a}}\right). \tag{43}$$

Then (J) also holds with R_a in place of R:

$$\lim_{a\to 0} \mathbb{P}\big(\mathrm{Cover}(\Lambda, R_a, a; M)\big) = \mathrm{e}^{-\mathrm{e}^{-\tau}}.$$

Proof. First, with the assumptions (J_1) and (J_2) of Janson's theorem [18, Lemma 8.1] fulfilled for R, the convergence (J) holds:

$$\lim_{a\to 0} \mathbb{P}\big(\mathrm{Cover}(\Lambda, R, a; M)\big) = \mathrm{e}^{-\mathrm{e}^{-\tau}}.$$

To prove the same for $\mathrm{Cover}(\Lambda, R_a, a; M)$, we use a coupling argument. Let $w > 0$ and $\eta := w/\log \frac{1}{a}$. By (42), it holds for all a sufficiently small that

$$\forall \rho > 0, \qquad \mathbb{P}(R_a > \rho) \leq \mathbb{P}\big((1+\eta)R > \rho\big).$$

Applying the inverse method, that is, considering the generalized inverses F_a^{-1} and F^{-1} of the distribution functions of R_a and R respectively, and setting $R_a := F_a^{-1}(U)$ and $R := F^{-1}(U)$ for a uniform variable U in $[0,1]$, we may then suppose for every small $a > 0$ that R_a and R are coupled so that

$$R_a \leq (1+\eta)R, \quad \text{almost surely.}$$

We then introduce independent, uniformly distributed variables X_i, $i \geq 1$, on M and, for every small $a > 0$, an independent Poisson-distributed variable N_a with mean $\Lambda(a) v_M(M)$, as well as a further independent family $(R_{a,i}, R_i)_{i\geq 1}$ of i.i.d. copies of (R_a, R). Thus, for every small $a > 0$, we have constructed a Poisson point process $\boldsymbol{\Xi}_a := \{X_i, R_{a,i}, R_i\}_{1\leq i\leq N_a}$ on $M \times (0,\infty)^2$ whose projections $\Xi_{1,a} := (X_i, R_{a,i})_{1\leq i\leq N_a}$ and $\Xi_{2,a} := (X_i, R_i)_{1\leq i\leq N_a}$ have intensity $\Lambda v_M(\mathrm{d}x) \otimes \mathbb{P}(R_a \in \mathrm{d}\rho)$ and $\Lambda v_M(\mathrm{d}x) \otimes \mathbb{P}(R \in \mathrm{d}\rho)$ respectively. In particular, with this construction the two covering events are given by

$$\mathrm{Cover}(\Lambda, R_a, a; M) = \left\{ \bigcup_{\Xi_{1,a}} B_M(X_i, aR_{a,i}) = M \right\} \text{ and } \mathrm{Cover}(\Lambda, R, a; M) = \left\{ \bigcup_{\Xi_{2,a}} B_M(X_i, aR_i) = M \right\}.$$

Since $R_{a,i} \le (1+\eta)R_i$ almost surely for all $i \ge 1$, we get

$$\begin{aligned}
\mathbb{P}\big(\mathrm{Cover}(\Lambda, R_a, a; M)\big) &= \mathbb{P}\left(\bigcup_{\Xi_{1,a}} B_M(X_i, aR_{a,i}) = M\right)\\
&\le \mathbb{P}\left(\bigcup_{\Xi_{2,a}} B_M(X_i, a(1+\eta)R_i) = M\right)\\
&= \mathbb{P}\big(\mathrm{Cover}(\Lambda, R, (1+\eta)a; M)\big),
\end{aligned}$$

and therefore,

$$\limsup_{a\to 0}\ \mathbb{P}\big(\mathrm{Cover}(\Lambda, R_a, a; M)\big) \le \limsup_{a\to 0}\ \mathbb{P}\big(\mathrm{Cover}(\Lambda^-, R, a; M)\big),$$

where, recalling $J(\Lambda, R, a; M) \to \tau$ by (J_2) and $\eta := w/\log\frac{1}{a}$,

$$\begin{aligned}
\Lambda^-(a) &:= \Lambda\left(\frac{a}{1+\eta}\right)\\
&= \frac{(1+\eta)^D}{b(R;M)a^D v_M(M)}\left(\log\frac{(1+\eta)^D}{b(R;M)a^D} + D\log\log\frac{(1+\eta)^D}{b(R;M)a^D} + \log\alpha(R) + \tau + o(1)\right)\\
&= \frac{1+D\eta+o(\eta)}{b(R;M)a^D v_M(M)}\left(\log\frac{1}{b(R;M)a^D} + D\log\log\frac{1}{b(R;M)a^D} + \log\alpha(R) + \tau + o(1)\right)\\
&= \left[1 + \frac{Dw}{\log\frac{1}{a}} + o\left(\frac{1}{\log\frac{1}{a}}\right)\right]\Lambda(a).
\end{aligned}$$

This entails $J(\Lambda^-, R, a; M) \to \tau - Dw$, so Janson's theorem applies with

$$\lim_{a\to 0}\ \mathbb{P}\big(\mathrm{Cover}(\Lambda^-, R, a; M)\big) = \mathrm{e}^{-\mathrm{e}^{-\tau - Dw}}.$$

Letting $w \to 0^+$, we have proved

$$\limsup_{a\to 0}\ \mathbb{P}\big(\mathrm{Cover}(\Lambda, R_a, a; M)\big) \le \mathrm{e}^{-\mathrm{e}^{-\tau}}.$$

To establish the other direction, we keep $w > 0$ and $\eta := \log\frac{1}{a}$, as well as the Poisson point process $\boldsymbol{\Xi}_a := \{X_i, R_{a,i}, R_i\}_{1\le i\le N_a}$ and its projections $\Xi_{1,a}$ and $\Xi_{2,a}$. It is known by the Kantorovich–Rubinstein theorem [10, Theorem 11.8.2] that

$$W_1(R_a, R) = \int_0^1 \left|F_a^{-1}(u) - F^{-1}(u)\right| \mathrm{d}u,$$

that is, with the same coupling of (R_a, R) constructed via the inverse method, $\mathbb{E}\,|R_a - R| = W_1(R_a, R)$. We then restrict $\boldsymbol{\Xi}_a$ by keeping only the radii $R_{a,i}$ such that $R_{a,i} \ge (1-\eta)R_i$:

$$\widetilde{\boldsymbol{\Xi}}_a := \Big\{(X_i, R_{a,i}, R_i) \in \boldsymbol{\Xi} : R_{a,i} \ge (1-\eta)R_i\Big\}.$$

Hence $\widetilde{\Xi}_{2,a} := \{(X_i, R_i) : (X_i, R_{a,i}, R_i) \in \widetilde{\boldsymbol{\Xi}}_a\} \subseteq \Xi_{2,a}$ has a smaller intensity, $\widetilde{\Lambda} v_M(\mathrm{d}x) \otimes \mathbb{P}(R_a \in \mathrm{d}\rho)$ with $\widetilde{\Lambda}(a) := \Lambda(a)\,\mathbb{P}(R_a \geq (1-\eta)R)$. Since, using ε_a fulfilling (43) and Markov's inequality,

$$\begin{aligned}
\mathbb{P}(R_a < (1-\eta)R) &\leq \mathbb{P}(R \leq \varepsilon_a) + \mathbb{P}(|R_a - R| > \eta\varepsilon_a) \\
&\leq \mathbb{P}(R \leq \varepsilon_a) + \frac{1}{w}\left(\varepsilon_a \log\frac{1}{a}\right) W_1(R_a, R) \\
&= o\left(\frac{1}{\log\frac{1}{a}}\right),
\end{aligned}$$

we have $\widetilde{\Lambda}(a) = \Lambda(a)[1 + o(\log^{-1}(\frac{1}{a}))]$, which by ($J_2$) implies that $J\big(\widetilde{\Lambda}, R, a; D\big) \to \tau$, i.e.,

$$\widetilde{\Lambda}(a) = \frac{1}{b(R;M)a^D v_M(M)}\left(\log\frac{1}{b(R;M)a^D} + D\log\log\frac{1}{b(R;M)a^D} + \log\alpha(R) + \tau + o(1)\right). \tag{$\widetilde{J}_2$}$$

Now,

$$\begin{aligned}
\mathbb{P}\big(\mathrm{Cover}(\Lambda, R_a, a; M)\big) &= \mathbb{P}\left(\bigcup_{\Xi_{1,a}} B_M(X_i, aR_{a,i}) = M\right) \\
&\geq \mathbb{P}\left(\bigcup_{\widetilde{\boldsymbol{\Xi}}_a} B_M(X_i, aR_{a,i}) = M\right) \\
&\geq \mathbb{P}\left(\bigcup_{\widetilde{\Xi}_{2,a}} B_M(X_i, a(1-\eta)R_i) = M\right) \\
&= \mathbb{P}\big(\mathrm{Cover}(\Lambda^+, R_i, (1-\eta)a; M)\big)
\end{aligned}$$

and therefore,

$$\liminf_{a\to 0}\ \mathbb{P}\big(\mathrm{Cover}(\Lambda, R_a, a; M)\big) \geq \liminf_{a\to 0}\ \mathbb{P}\big(\mathrm{Cover}(\Lambda^+, R, a; M)\big),$$

where, recalling ($\widetilde{J}_2$) and $\eta := w/\log\frac{1}{a}$,

$$\begin{aligned}
\Lambda^+(a) &:= \widetilde{\Lambda}\left(\frac{a}{1-\eta}\right) \\
&= \frac{(1-\eta)^D}{b(R;M)a^D v_M(M)}\left(\log\frac{(1-\eta)^D}{b(R;M)a^D} + D\log\log\frac{(1-\eta)^D}{b(R;M)a^D} + \log\alpha(R) + \tau + o(1)\right) \\
&= \frac{1 - D\eta + o(\eta)}{b(R;M)a^D v_M(M)}\left(\log\frac{1}{b(R;M)a^D} + D\log\log\frac{1}{b(R;M)a^D} + \log\alpha(R) + \tau + o(1)\right) \\
&= \left[1 - \frac{Dw}{\log\frac{1}{a}} + o\left(\frac{1}{\log\frac{1}{a}}\right)\right]\Lambda(a).
\end{aligned}$$

This entails $J(\Lambda^+, R, a; M) \to \tau - Dw$, so Janson's theorem applies again with

$$\lim_{a\to 0}\ \mathbb{P}\big(\mathrm{Cover}(\Lambda^+, R, a; M)\big) = \mathrm{e}^{-\mathrm{e}^{-\tau + Dw}}.$$

It remains to let $w \to 0^+$ to complete the proof. □

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