Variance conjecture in Schatten balls

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Phenomena in High Dimension
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Outline

Introduction and results

Proof summary of Theorems A and B

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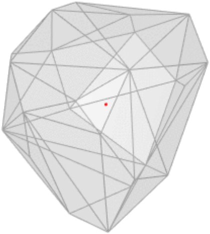

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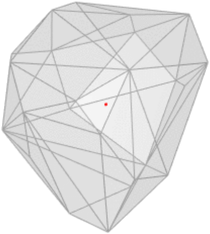

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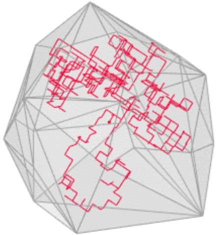

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Hence for n large, $\operatorname{Var}_K |\boldsymbol{x}|^2 \leq \frac{1}{2} \cdot \frac{\big(\operatorname{Tr} \Sigma_K\big)^2}{d_n} \leq \frac{1}{2} \|\Sigma_K\|_{\mathsf{op}} \operatorname{Tr} \Sigma_K$.

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- Case $p = \infty$ proved by Radke and Vritsiou (2020).

General statements

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For either $E = \mathcal{M}_n(\mathbb{F})$ or $E = \{T = T^*\}$, for any $p \in [1, \infty)$ and $q > 0$,

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Theorem B (Asymptotics of the q-inertia moments).
For $E = \{T = T^*\}$, $p \in (3, \infty)$ and any $q > 0$,

$$\frac{I_q(K)}{I_2(K)} = 1 + \frac{(q-2)(p-2)^2}{16p^2\,d_n} + o\left(\frac{1}{d_n}\right).$$

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Theorem 1 follows by taking $q = 4$ and raising to the fourth power.

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$$\overset{T=USV^*}{=} \frac{c_n}{\left|B_E\left(S_p^n\right)\right|} \int_{\|\boldsymbol{s}\|_p \leq 1} |\boldsymbol{s}|^q f_{a,b,c}(\boldsymbol{s}) \,\mathrm{d}\boldsymbol{s}$$

where c_n is explicit and depend on $|U_n(\mathbb{F})|$ and E, and

$$f_{a,b,c}(\boldsymbol{s}) := \prod_{1 \leq i < j \leq n} |s_i^a - s_j^a|^b \prod_{i=1}^{n} |s_i|^c,$$

with $b := \dim_{\mathbb{R}} \mathbb{F}$, $a \in \{1,2\}$, $c \in \{0,1,3\}$ depending on E and $\mathbb{F}$.

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$$\overset{T=USV^*}{=} \frac{c_n}{\left|B_E\left(S_p^n\right)\right|} \int_{\|\boldsymbol{s}\|_p \leq 1} |\boldsymbol{s}|^q \, f_{a,b,c}(\boldsymbol{s}) \,\mathrm{d}\boldsymbol{s}$$

where c_n is explicit and depend on $|U_n(\mathbb{F})|$ and E, and

$$f_{a,b,c}(\boldsymbol{s}) := \prod_{1 \leq i < j \leq n} |s_i^a - s_j^a|^b \prod_{i=1}^{n} |s_i|^c,$$

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Convexity trick. $\times\, \Gamma\left(1 + \frac{d_n + q}{p}\right)$, then $\boldsymbol{s} \leftarrow t^{-\frac{1}{p}} \boldsymbol{x}$, and last Fubini:

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- Saint-Raymond '84: formulas and bounds for $|B_E(S_p^n)|$.

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- Kabluchko-Prochno-Thäle '19,'20: $|B_E(S_p^n)|^{1/d_n}$ & WLLN for $\boldsymbol{s}(T)$.

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We introduce

$$\mathbb{P}_{n,p}(\mathrm{d}\boldsymbol{x}) := \frac{1}{Z_{n,p}} \, f_{a,b,c}(\boldsymbol{x}) \, \mathrm{e}^{-abn\gamma_p \|\boldsymbol{x}\|_p^p} \, \mathrm{d}\boldsymbol{x}. \qquad \left(\gamma_p := \tfrac{1}{2} B\left(\tfrac{p}{2}, \tfrac{1}{2}\right)\right)$$

Reduction to linear statistics of β-ensembles

$$\mathbb{E}_K\,|\boldsymbol{x}|^q = \frac{Z_{n,p}\,c_n(abn\gamma_p)^{\frac{d_n+q}{p}}}{\left|B_E\left(S_p^n\right)\right|\Gamma\left(1+\frac{d_n+q}{p}\right)}\int_{\mathbb{R}^n}|\boldsymbol{x}|^q\,\mathbb{P}_{n,p}(\mathrm{d}\boldsymbol{x}). \qquad (L_q)$$

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- $\dfrac{(L_q)^{\frac{1}{q}} \cdot (L_0)^{-\frac{1}{d_n} - \frac{1}{q}}}{\sqrt{d_n} \left|B_E\left(S_p^n\right)\right|^{\frac{1}{d_n}}}$ gives

$$\frac{I_q(K)}{\sqrt{d_n}} = \frac{\Gamma\left(1 + \frac{d_n}{p}\right)^{\frac{1}{q} + \frac{1}{d_n}}}{\Gamma\left(1 + \frac{d_n+q}{p}\right)^{\frac{1}{q}} {c_n}^{\frac{1}{d_n}}} \cdot \frac{\left(\mathbb{E}_{n,p} \, |\boldsymbol{x}|^q\right)^{\frac{1}{q}}}{{Z_{n,p}}^{\frac{1}{d_n}}}.$$

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$$\mathbb{E}_K |\boldsymbol{x}|^q = \frac{Z_{n,p}\, c_n (abn\gamma_p)^{\frac{d_n+q}{p}}}{\left|B_E\left(S_p^n\right)\right| \Gamma\left(1+\frac{d_n+q}{p}\right)} \int_{\mathbb{R}^n} |\boldsymbol{x}|^q \, \mathbb{P}_{n,p}(\mathrm{d}\boldsymbol{x}). \qquad (L_q)$$

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✓

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✓

First order asymptotics

Let $V_p(x) := 2\gamma_p |x|^p$ and $L_n(\boldsymbol{x}) := n^{-1} \sum_{i=1}^{n} \delta_{x_i}$.

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Suppose $E = \{T = T^*\}$. In this case $a = 1$, $b = \dim_{\mathbb{R}} \mathbb{F}$, $c = 0$, so

$$\begin{aligned}\mathbb{P}_{n,p}(\mathrm{d}\boldsymbol{x}) &= \frac{1}{Z_{n,p}} \prod_{1 \le i < j \le n} |x_i - x_j|^b \, \mathrm{e}^{-bn\gamma_p \|\boldsymbol{x}\|_p^p} \, \mathrm{d}\boldsymbol{x} \\ &= \frac{1}{Z_{n,p}} \, \mathrm{e}^{-\frac{bn^2}{2} H_{n,p}(\boldsymbol{x})} \, \mathrm{d}\boldsymbol{x},\end{aligned}$$

with $\quad H_{n,p}(\boldsymbol{x}) := \frac{1}{n} \sum_{i=1}^n V_p(x_i) - \frac{1}{n^2} \sum_{i \neq j} \log |x_i - x_j|$

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$$\mathcal{I}_p(\mu) := \int_{\mathbb{R}} V_p(x)\,\mu(\mathrm{d}x) - \iint_{\mathbb{R}^2_{\neq}} \log|x-y|\,\mu(\mathrm{d}x)\mu(\mathrm{d}y).$$

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minimal when $L_n(\boldsymbol{x}) \in \operatorname*{argmin}_{\mathcal{P}_1(\mathbb{R})} \mathcal{I}_p$

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- $\operatorname*{argmin}_{\mathcal{P}_1(\mathbb{R})} \mathcal{I}_p = \{\mu_p\}$, $\mu_p \ll \mathrm{Leb}$ and $\mathrm{Supp}\,\mu_p = [-1,1]$;
- $\mathbb{E}_{n,p}\langle L_n(\boldsymbol{x}), f\rangle \xrightarrow[n\to\infty]{} \langle \mu_p, f\rangle$ for all $f \in \mathcal{C}_b$;
- $-\frac{1}{d_n}\log Z_{n,p} \xrightarrow[n\to\infty]{} \mathcal{I}_p(\mu_p) = \log 2 + \frac{3}{2p}$.

[Saff-Totik '97,
Ben Arous-Guionnet '97,
Johansson '98,
Hiai-Petz '00]

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Let $L_{n,p}$ be a random measure having same law as $\dfrac{1}{n}\sum_{i=1}^{n}\delta_{x_i}$ under $\mathbb{P}_{n,p}$.

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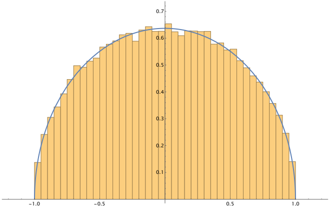
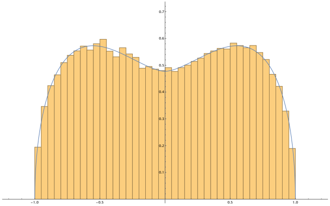
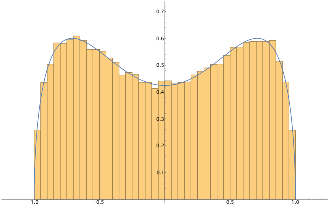


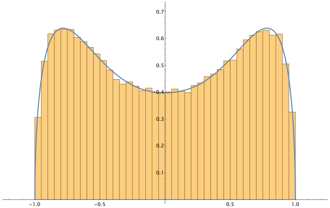
(A) $p = 2$;



(B) $p = 3$;



(C) $p = 4$;



(D) $p = 5$.

First order asymptotics

Let $L_{n,p}$ be a random measure having same law as $\frac{1}{n}\sum_{i=1}^{n}\delta_{x_i}$ under $\mathbb{P}_{n,p}$.

Lemma. There exist $B, c, C > 0$ such that, for $U := (-B, B)$,

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and Theorem A follows ($\langle \mu_p, x^2\rangle = \frac{p}{2p+4}$ using the density of μ_p).

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and Theorem A follows ($\langle \mu_p, x^2\rangle = \frac{p}{2p+4}$ using the density of μ_p).
- As for Theorem B, at this point we can only conclude that
$$\lim_{n\to\infty} \frac{I_q\big(B_E\big(S_p^n\big)\big)}{I_2\big(B_E\big(S_p^n\big)\big)} = 1.$$

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$$F_{n,p} := n\Big(\langle L_{n,p}, x^2 \rangle - \langle \mu_p, x^2 \rangle\Big)?$$

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Theorem C. Suppose $E = \{T = T^*\}$ and $p \in (3, \infty)$. As $n \to \infty$, $F_{n,p}$ converges in law and in moments to some $\mathcal{N}(m_p, \frac{1}{4b})$ variable.

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We deduce that

$$\frac{\mathbb{E}_{n,p}\,|\boldsymbol{x}|^q}{\left(\mathbb{E}_{n,p}\,|\boldsymbol{x}|^2\right)^{\frac{q}{2}}} = \frac{\mathbb{E}\,\langle L_{n,p}, x^2\rangle^{\frac{q}{2}}}{\left(\mathbb{E}\,\langle L_{n,p}, x^2\rangle\right)^{\frac{q}{2}}} = 1 + \frac{q(q-2)}{8\langle \mu_p, x^2\rangle^2} \cdot \frac{\operatorname{Var} F_{n,p}}{n^2} + o\Big(\frac{1}{n^2}\Big).$$

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Theorem B follows by raising to $\frac{1}{q}$ and replacing $\operatorname{Var} F_{n,p}$ and $\langle \mu_p, x^2\rangle$.

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- Taylor-expand as $t \to 0$ ($n \to \infty$) and choose $\psi_p \in \mathcal{C}^3$ solution to

$$V_p'(x)\psi(x) - 2\int \frac{\psi(x) - \psi(y)}{x - y}\,\mu_p(\mathrm{d}y) = x^2 + c, \qquad x \in \mathbb{R}.$$

Thanks for your attention!