ASYMPTOTICS OF THE INERTIA MOMENTS AND THE VARIANCE CONJECTURE IN SCHATTEN BALLS

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Abstract. We study the first and second orders of the asymptotic expansion, as the dimension goes to infinity, of the moments of the Hilbert-Schmidt norm of a uniformly distributed matrix in the p-Schatten unit ball. We consider the case of matrices with real, complex or quaternionic entries, self-adjoint or not. When $p > 3$, this asymptotic expansion allows us to establish a generalized version of the variance conjecture for the family of p-Schatten unit balls of self-adjoint matrices.

1. Introduction

Let $\mathbb{F} := \mathbb{R}$, $\mathbb{C}$, or the quaternionic field $\mathbb{H}$. For $n \geq 1$, we work on $\mathbb{F}^n$ which is seen as a βn-dimensional vector space over $\mathbb{R}$, with $\beta \in \{1, 2, 4\}$. We generically denote by E either the space $\mathcal{M}_n(\mathbb{F})$ of $n \times n$ matrices with entries from the field $\mathbb{F}$, or the subspace of self-adjoint matrices in $\mathcal{M}_n(\mathbb{F})$. We view E as a vector space over $\mathbb{R}$, whose dimension d_n depends on n, $\mathbb{F}$, and whether or not we impose self-adjointness (see (6) below for the exact formula). In any case, we equip E with the Euclidean structure defined by the Hilbert-Schmidt norm $\|T\|_{\mathrm{HS}} := (\mathrm{tr}(T^*T))^{1/2}$, and with the Lebesgue measure denoted by $\mathrm{d}T$. We write $|A|$ for the Lebesgue measure of any Borel set $A \subseteq E$, when it is finite.

For every $z \in \mathbb{F}^n$, let

$$\|z\|_p := \left(\sum_{i=1}^n |z_i|^p\right)^{\frac{1}{p}} \quad \text{for } 1 \leq p < \infty, \quad \text{and } \|z\|_\infty := \max_{1\leq i\leq n} |z_i|.$$

For any matrix $T \in \mathcal{M}_n(\mathbb{F})$, let $s(T) := (s_1(T), \ldots, s_n(T)) \in \mathbb{R}^n$ be the tuple of singular values of T, i.e., the eigenvalues of $\sqrt{T^*T}$. For every $1 \leq p \leq \infty$, let $\sigma_p(T) = \|s(T)\|_p$: this defines a norm on $\mathcal{M}_n(\mathbb{F})$, called the p-Schatten norm (see [10, Chapter IV]), and the corresponding normed space is denoted by S_p^n, with unit ball

$$B(S_p^n) := \{T \in \mathcal{M}_n(\mathbb{F}) : \sigma_p(T) \leq 1\}.$$

In the special case $p = 2$, $\sigma_2(T) = \|T\|_{\mathrm{HS}}$ and we recover the Euclidean structure.

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Let us note that, when E is a space of self-adjoint matrices, all matrices in E are diagonalizable, and $\sigma_p(T)$ is then the p-norm of the vector $(\lambda_1(T), \dots, \lambda_n(T))$ of eigenvalues of T. Taking random Gaussian entries for T then leads to the classical orthogonal (GOE), unitary (GUE) and symplectic (GSE) ensembles in random matrix theory, see for instance [32, Chapter 2].

The purpose of this paper is to study the first and second order in the asymptotic expansion, as n goes to infinity, of the q-inertia moment

$$I_q(B_E(S_p^n)) := \frac{\left(\mathbb{E}\,\|T\|_{\mathrm{HS}}^q\right)^{1/q}}{|B_E(S_p^n)|^{1/d_n}}, \tag{1}$$

for T uniformly distributed in $B_E(S_p^n) := B(S_p^n) \cap E$. These quantities appear in various conjectures in asymptotic geometric analysis that we now present.

1.1. **Relevant conjectures in asymptotic geometric analysis.** The interplay between the classical conjectures in asymptotic geometric analysis that are the hyperplane conjecture, the Kannan-Lovász-Simonovits conjecture and the variance conjecture, is quite intricate; we state them here briefly to provide context and refer the interested reader to [1, 12, 25, 31] for an in-depth presentation of the field.

Let K be a symmetric convex body in $\mathbb{R}^d$. The isotropic constant of K is defined by

$$L_K := \min_{\det(T)=1} \frac{1}{\sqrt{d}} \left(\frac{1}{|K|^{1+\frac{2}{d}}} \int_K \|T\mathbf{x}\|_2^2 \, \mathrm{d}\mathbf{x} \right)^{\frac{1}{2}}, \tag{2}$$

and its covariance matrix Σ is defined by

$$\Sigma_{i,j} := \frac{1}{|K|} \int_K x_i x_j \, \mathrm{d}\mathbf{x}, \quad 1 \le i, j \le d.$$

We say that K is in isotropic position when $\Sigma = \mathrm{Id}$, in which case $L_K = \frac{1}{|K|^{1/d}}$ and the minimum in Equation (2) is attained for $T = \mathrm{Id}$. The hyperplane conjecture asks for a uniform upper bound on L_K for all symmetric convex bodies and dimensions d, see [25, 33]. The Kannan-Lovász-Simonovits (KLS) conjecture [23] asks for the existence of a universal constant C such that, for every dimension d, every random vector X uniformly distributed on a symmetric convex body $K \subset \mathbb{R}^d$, and every smooth function f on $\mathbb{R}^d$,

$$\mathrm{Var}\, f(X) \le C \max_{\theta \in S^{d-1}} \mathbb{E}\langle X, \theta\rangle^2 \cdot \mathbb{E}\,\|\nabla f(X)\|_2^2. \tag{3}$$

This conjecture is satisfied for various families of convex bodies [1], like the unit balls B_p^n of the classical ℓ_p^n spaces, after [39], or the Orlicz balls after [6, 26]. In a recent breakthrough, Yuansi Chen [13], using methods introduced by Eldan in [16] and developed in [30], proved that inequality (3) is valid for a value C still depending on the dimension but smaller than any power of d, $C = d^{o_d(1)}$. Specifying the KLS conjecture to $f(x) = \|x\|_2^2$ one gets the following weaker conjecture.

Generalized variance conjecture. *There is a constant $C \geq 1$ such that for every dimension d and for every symmetric convex body $K \subset \mathbb{R}^d$, we have*

$$\operatorname{Var} \|X\|_2^2 \leq C \max_{\theta \in S^{d-1}} \mathbb{E}\langle X, \theta\rangle^2 \cdot \mathbb{E}\,\|X\|_2^2 = C\|\Sigma\| \cdot \operatorname{tr}(\Sigma) \tag{4}$$

where X is uniformly distributed on K and Σ is its covariance matrix.

Klartag [24] proved this conjecture for every unconditional convex bodies, that means convex bodies which are invariant under coordinate hyperplane reflections, while Barthe and Cordero [5] studied these questions for convex bodies with more general symmetries.

These conjectures were studied in the particular case of the unit balls of the p-Schatten normed spaces. König, Meyer and Pajor [27] established the hyperplane conjecture for these families of particular bodies. Guédon and Paouris [17] proved a concentration of the volume through the study of the moments of the Hilbert-Schmidt norm of a random matrix uniformly distributed in $B_E(S_p^n)$. This method was generalized by Radke and Vritsiou [35] and Vritsiou [41] to prove the generalized variance conjecture when the space of matrices is equipped with the operator norm, that is the case $p = \infty$.

1.2. **Results.** We start by stating a result on the first order in the asymptotic expansion of $I_q(B_E(S_p^n))$. We give the exact computation of the limit of $I_q(B_E(S_p^n))$, correctly normalized by the square root of the dimension of E as a vector space over $\mathbb{R}$. This dimension will be denoted by d_n.

Theorem 1.1 (Limit of the normalized inertia)**.** *Let E be $\mathcal{M}_n(\mathbb{F})$ or its restriction to self-adjoint matrices, equipped with the p-Schatten norm for some $p \in [1, \infty)$. Then for every $q > 0$,*

$$\frac{I_q(B_E(S_p^n))}{\sqrt{d_n}} = \frac{\left(\mathbb{E}\,\|T\|_{\mathrm{HS}}^q\right)^{1/q}}{\sqrt{d_n}\,|B_E(S_p^n)|^{1/d_n}} \xrightarrow[n\to\infty]{} \mathrm{e}^{\frac{1}{2p}-\frac{3}{4}} \sqrt{\frac{p}{\pi(p+2)}}, \tag{5}$$

where T is uniformly distributed in $B_E(S_p^n)$.

In the self-adjoint case, we note that this result can be established as a consequence of a weak law of large numbers proved by Kabluchko, Prochno and Thäle, [21, Theorem 4.7] combined with a truncation argument [34, Theorem 11.1.2].

The general case is of particular interest, as it allows for instance, for $q = 2$ and $E = \mathcal{M}_n(\mathbb{F})$, a computation of the limit of the isotropic constant of the unit ball S_p^n. Indeed, in this (and only in this) non self-adjoint case, the Schatten unit balls are in isotropic position [4, 27, 35], so that the left-hand side of (5) is the isotropic constant defined by (2), and Theorem 1.1 gives the precise value of its limit as n goes to infinity.

Our main result establishes, in the self-adjoint case, the second term in the asymptotic expansion of the quotient $I_q(B_E(S_p^n))/I_2(B_E(S_p^n))$.

Theorem 1.2 (Asymptotic expansion of the q-inertia moment)**.** *Let E be the space of real symmetric, complex Hermitian or Hermitian quaternionic matrices equipped with the p-Schatten norm for some $p \in (3, \infty)$. Then for every $q > 0$,*

$$\frac{I_q(B_E(S_p^n))}{I_2(B_E(S_p^n))} = 1 + \frac{(q-2)(p-2)^2}{16p^2\, d_n} + o\left(\frac{1}{d_n}\right)$$

as n goes to infinity, where I_q is defined in (1).

As a corollary, we give a positive answer to the generalized variance conjecture for these families of convex bodies.

Corollary 1.3 (Variance conjecture in Schatten balls)**.** *Let $p \in (3, \infty)$ and E be the space of real symmetric, complex Hermitian or Hermitian quaternionic matrices equipped with the Schatten p-norm. Then, for all n large enough,*

$$\mathrm{Var}(\|T\|_{\mathrm{HS}}^2) \leq \frac{1}{2}\,\|\Sigma\| \cdot \mathrm{tr}(\Sigma),$$

where T is uniformly distributed in $B_E(S_p^n)$ and Σ is the covariance matrix of $B_E(S_p^n)$.

Indeed, specializing Theorem 1.2 to $q = 4$, we get

$$\lim_{n\to\infty} d_n \frac{\mathrm{Var}(\|T\|_{\mathrm{HS}}^2)}{\left(\mathbb{E}\,\|T\|_{\mathrm{HS}}^2\right)^2} = \frac{(p-2)^2}{2p^2} < \frac{1}{2}.$$

We conclude using that $\mathbb{E}\,\|T\|_{\mathrm{HS}}^2 = \mathrm{tr}(\Sigma) \leq d_n\,\|\Sigma\|$.

1.3. **Strategy of proof.** Recall that E is either $\mathcal{M}_n(\mathbb{F})$ or the subspace of self-adjoint matrices in $\mathcal{M}_n(\mathbb{F})$. Using a strategy developed by Saint-Raymond [37], König, Meyer and Pajor [27], and pushed further by Guédon and Paouris [17], Radke and Vritsiou [35] and Kabluchko, Prochno and Thäle [21], we first reduce the computations of $I_q(B_E(S_p^n))$ to integrals over $\mathbb{R}^n$ with respect to a Gibbs probability measure $\mathbb{P}_{n,p}$. This distribution governs a so called β-ensemble consisting of n unit charges interacting through a logarithmic Coulomb potential and confined with an external potential, see (12). Kabluchko, Prochno and Thäle [20, 21, 22] have shown that this connection with linear statistics of β-ensembles allows to establish, when the dimension tends to infinity, exact asymptotics for the volume of the Schatten unit balls as well as a weak law of large number for the joint law of the eigenvalues/singular values.

More precisely, defining $L_{n,p} := n^{-1}\sum_{i=1}^n \delta_{x_i}$ to be the random empirical measure under $\mathbb{P}_{n,p}$, see (12), we recall in Lemma 2.2 that the computation of $I_q(B_E(S_p^n))$ can be expressed in terms of the normalization constant $Z_{n,p}$ appearing in the definition of $\mathbb{P}_{n,p}$, and the moment $\mathbb{E}\,\langle L_{n,p}, h_2\rangle^{q/2}$ of the linear statistics associated with the function $h_2(x) := x^2$. Thus, the task essentially amounts to estimating asymptotics for the normalizing constant $Z_{n,p}$ and for these linear statistics associated with the function h_2. Some of these asymptotics are now well understood in the literature on random matrices [9, 18, 19, 34]. Our proof of Theorem 1.1 clarifies the connection between the first order estimates of $I_q(B(S_p^n) \cap E)$ and the convergence of the empirical distribution towards the equilibrium measure due to Johansson [19] that we state in Lemma 4.2,

or Hiai-Petz [18], see Lemma 4.6. To prove Theorem 1.2, we need to understand more precisely the fluctuations of the linear statistics $\langle L_{n,p}, h_2\rangle$ in the self-adjoint case. We rely here on recent results of Bekerman, Leblé, Serfaty [8] (see also Lambert, Ledoux, Webb [29]) concerning the CLT for fluctuations of β-ensembles with general potential. Let us note that in these results, the regularity of the external potential V in the Gibbs measure $\mathbb{P}_{n,p}$ plays a prominent role. It was assumed to be a polynomial of even degree with positive leading coefficient in the seminal paper [19]. This condition has then been relaxed over the last two decades to include real-analytic potentials [11, 28, 38] and, more recently, potentials of class $\mathcal{C}^r$ with r reasonably large [7, 8, 29]. However we are working in a very specific case where $V(x)$ is proportional to $|x|^p$. Proposition 4.3 states that the result of [8] applies when $p > 3$. It is proved by tracing back in the proof of [8] various places where the regularity of V is needed, in particular to prove regularity of the solution to a key master equation.

1.4. **Organization of the paper.** In Section 2, we recall the formulas relating the q-inertia moment I_q, the normalizing constant $Z_{n,p}$, and the expected q-moment of the linear statistics $\langle L_{n,p}, h_2\rangle$. We gather, in Section 3, some uniform moment bounds which are needed to accommodate the fact that we study a linear statistics for the unbounded function h_2. This generalizes a classical truncation argument to all possible parameters in the density of $\mathbb{P}_{n,p}$. The proof of our main results is done in Section 4. We postpone to Section 5 the discussion on technical properties of the equilibrium measure and on the regularity of the solution to the master equation.

2. Reduction to integrals over $\mathbb{R}^n$

A function $F\colon \mathbb{R}^n \to \mathbb{R}$ is said to be symmetric if for every $x \in \mathbb{R}^n$ and every permutation π on $\{1,\dots,n\}$ we have $F(x_1,\dots,x_n) = F(x_{\pi(1)},\dots,x_{\pi(n)})$. Let E be $\mathcal{M}_n(\mathbb{F})$ or its restriction to self-adjoint matrices. The following change of variable formula [2, Propositions 4.1.1 & 4.1.3] makes a connection between the Schatten unit ball $B_E(S_p^n)$ and the classical unit ℓ_p^n-ball $B_p^n \subset \mathbb{R}^n$: for every symmetric, continuous function $F\colon \mathbb{R}^n \to \mathbb{R}$,

$$\int_{B_E(S_p^n)} F\big(s(T)\big)\,\mathrm{d}T = c_n \int_{B_p^n} F(\mathbf{x})\, f_{a,b,c}(\mathbf{x})\mathrm{d}\mathbf{x},$$

where c_n is a positive constant,

$$f_{a,b,c}(\mathbf{x}) \;:=\; \prod_{1\le i<j\le n} \big|x_i^a - x_j^a\big|^b \cdot \prod_{i=1}^n |x_i|^c, \qquad \mathbf{x} := (x_1,\dots,x_n) \in \mathbb{R}^n,$$

defines a positively homogeneous function of degree $d_n - n$, with

$$d_n := \dim_{\mathbb{R}} E = ab\,\frac{n(n-1)}{2} + (c+1)n \tag{6}$$

being the dimension of the subspace E over $\mathbb{R}$. The constant c_n is explicit and related to the volume of the unitary group $U_n(\mathbb{F})$; it depends only on a, b and n, and its exact and asymptotic values are known and given in Lemma 4.7:

Matrix Ensemble	$\mathbb{F}$	a	b	c
real	$\mathbb{R}$	2	1	0
complex	$\mathbb{C}$	2	2	1
quaternion	$\mathbb{H}$	2	4	3
real symmetric	$\mathbb{R}$	1	1	0
complex Hermitian	$\mathbb{C}$	1	2	0
Hermitian quaternionic	$\mathbb{H}$	1	4	0

TABLE 1. Possible choices for E, and corresponding parameters.

In all cases, $b = \beta = \dim_{\mathbb{R}}(\mathbb{F})$. In the self-adjoint cases, $a = 1$ and $c = 0$. In the cases where $E = \mathcal{M}_n(\mathbb{F})$, the computations depend on the singular values, $a = 2$ and $c = \beta - 1$. See [2, Chapter 4] for details.

$$\sqrt{n} \cdot c_n^{1/d_n} \sim \mathrm{e}^{\frac{3}{4}} \sqrt{\frac{4\pi}{ab}} \quad \text{as } n \to \infty. \tag{7}$$

Combining this change of variables with a classical trick in convexity leads to the following expression.

Lemma 2.1 (Change of variables, [17])**.** *For every symmetric, continuous, positively homogeneous function $F\colon \mathbb{R}^n \to \mathbb{R}$ of degree $k \geq 0$, one has*

$$\int_{B_E(S_p^n)} F\big(s(T)\big)\,\mathrm{d}T = \frac{c_n}{\Gamma\Big(1 + \frac{d_n+k}{p}\Big)} \int_{\mathbb{R}^n} F(\mathbf{x})\,\mathrm{e}^{-\|\mathbf{x}\|_p^p}\, f_{a,b,c}(\mathbf{x})\,\mathrm{d}\mathbf{x},$$

where Γ is Euler's Gamma function.

We refer to [17] for the details of the computations. Let

$$\gamma_p := \frac{\Gamma\big(\frac{p}{2}\big)\Gamma\big(\frac{1}{2}\big)}{2\Gamma\big(\frac{p+1}{2}\big)}. \tag{8}$$

Applying Lemma 2.1 with $F \equiv 1$ which is homogeneous of degree 0 yields, after a straightforward change of variables,

$$\big|B_E(S_p^n)\big| = (abn\gamma_p)^{\frac{d_n}{p}} \cdot \frac{c_n}{\Gamma\Big(1 + \frac{d_n}{p}\Big)} \cdot Z_{n,p}, \tag{9}$$

where

$$Z_{n,p} := \int_{\mathbb{R}^n} \mathrm{e}^{-abn\gamma_p\|\mathbf{x}\|_p^p}\, f_{a,b,c}(\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{10}$$

Using now Lemma 2.1 with $F(\mathbf{x}) := \big(\sum_{i=1}^n x_i^2\big)^{q/2}$ which is homogeneous of degree q, we obtain, for T uniformly distributed in $B_E(S_p^n)$,

$$\mathbb{E}\,\|T\|_{\mathrm{HS}}^q = (abn\gamma_p)^{\frac{q}{p}} \cdot \frac{\Gamma\Big(1 + \frac{d_n}{p}\Big)}{\Gamma\Big(1 + \frac{d_n+q}{p}\Big)} \cdot n^{\frac{q}{2}} \int_{\mathbb{R}^n} \left(\frac{1}{n}\sum_{i=1}^n x_i^2\right)^{\frac{q}{2}} \mathbb{P}_{n,p}(\mathrm{d}\mathbf{x}), \tag{11}$$

where $\mathbb{P}_{n,p}(\mathrm{d}\mathbf{x})$ is the probability measure

$$\mathbb{P}_{n,p}(\mathrm{d}\mathbf{x}) := \frac{1}{Z_{n,p}} \cdot f_{a,b,c}(\mathbf{x})\, \mathrm{e}^{-abn\gamma_p \|\mathbf{x}\|_p^p}\, \mathrm{d}\mathbf{x}, \qquad \mathbf{x} \in \mathbb{R}^n. \tag{12}$$

We conclude this section by expressing the quantities appearing in Theorems 1.1 and 1.2 in terms of the integral of $h_2(x) := x^2$ with respect to the random empirical measure $L_{n,p} := n^{-1} \sum_{i=1}^n \delta_{x_i}$ under $\mathbb{P}_{n,p}$.

Lemma 2.2 (From inertia to β-ensembles)**.** *For every $q > 0$, we have as $n \to \infty$,*

$$\frac{1}{\sqrt{d_n}}\, I_q(B_E(S_p^n)) \sim \frac{\mathrm{e}^{-\frac{1}{p}-\frac{3}{4}}}{\sqrt{2\pi}}\, Z_{n,p}^{-1/d_n} \left(\mathbb{E}\, \langle L_{n,p}, h_2 \rangle^{\frac{q}{2}} \right)^{\frac{1}{q}} \tag{13}$$

and

$$\frac{I_q(B_E(S_p^n))}{I_2(B_E(S_p^n))} = \left(1 - \frac{q-2}{abp\, n^2} + o\left(\frac{1}{n^2}\right)\right) \frac{\left(\mathbb{E}\, \langle L_{n,p}, h_2 \rangle^{q/2}\right)^{1/q}}{\left(\mathbb{E}\, \langle L_{n,p}, h_2 \rangle\right)^{1/2}}. \tag{14}$$

Proof. Observe that

$$\int_{\mathbb{R}^n} \left(\frac{1}{n} \sum_{i=1}^n x_i^2 \right)^{\frac{q}{2}} \mathbb{P}_{n,p}(\mathrm{d}\mathbf{x}) = \mathbb{E}\, \langle L_{n,p}, h_2 \rangle^{\frac{q}{2}}.$$

By combining (11) and (9), the quantity appearing in Theorem 1.1 is

$$\begin{aligned}
\frac{1}{\sqrt{d_n}}\, I_q(B_E(S_p^n)) &= \frac{1}{\sqrt{d_n}} \frac{\left(\mathbb{E}\, \|T\|_{\mathrm{HS}}^q\right)^{1/q}}{|B_E(S_p^n)|^{1/d_n}} \\
&= \frac{1}{\sqrt{d_n}} \left(\frac{\Gamma\left(1 + \frac{d_n}{p}\right)}{\Gamma\left(1 + \frac{d_n+q}{p}\right)} \right)^{\frac{1}{q}} \frac{\Gamma\left(1 + \frac{d_n}{p}\right)^{\frac{1}{d_n}}}{c_n^{1/d_n}\, Z_{n,p}^{1/d_n}} \sqrt{n} \left(\mathbb{E}\, \langle L_{n,p}, h_2 \rangle^{\frac{q}{2}} \right)^{\frac{1}{q}}, \\
&\sim \sqrt{\frac{2}{abn}}\, \mathrm{e}^{-\frac{1}{p}}\, c_n^{-1/d_n} Z_n^{-1/d_n} \left(\mathbb{E}\, \langle L_{n,p}, h_2 \rangle^{\frac{q}{2}} \right)^{\frac{1}{q}},
\end{aligned}$$

using that $d_n \sim abn^2/2$ as $n \to \infty$, $\Gamma(1+x)^{\frac{1}{x}} \sim x/\mathrm{e}$ and $\Gamma(1+x+\alpha)/\Gamma(1+x) \sim x^\alpha$ as $x \to \infty$, for any fixed number α. Thus, by the asymptotics of c_n in (7),

$$\frac{1}{\sqrt{d_n}}\, I_q(B_E(S_p^n)) \sim \frac{\mathrm{e}^{-\frac{1}{p}-\frac{3}{4}}}{\sqrt{2\pi}}\, Z_{n,p}^{-1/d_n} \left(\mathbb{E}\, \langle L_{n,p}, h_2 \rangle^{\frac{q}{2}} \right)^{\frac{1}{q}}.$$

Regarding the quantity involved in Theorem 1.2, two applications of (11) give

$$\begin{aligned}
\frac{I_q(B_E(S_p^n))}{I_2(B_E(S_p^n))} &= \frac{\left(\mathbb{E}\, \|T\|_{\mathrm{HS}}^q\right)^{1/q}}{\left(\mathbb{E}\, \|T\|_{\mathrm{HS}}^2\right)^{1/2}} \\
&= \left(\frac{\Gamma\left(1 + \frac{d_n+q}{p}\right)}{\Gamma\left(1 + \frac{d_n}{p}\right)} \right)^{-\frac{1}{q}} \cdot \left(\frac{\Gamma\left(1 + \frac{d_n+2}{p}\right)}{\Gamma\left(1 + \frac{d_n}{p}\right)} \right)^{\frac{1}{2}} \cdot \frac{\left(\mathbb{E}\, \langle L_{n,p}, h_2 \rangle^{q/2}\right)^{1/q}}{\left(\mathbb{E}\, \langle L_{n,p}, h_2 \rangle\right)^{1/2}}.
\end{aligned}$$

It is also well known [40, (3.23), p. 62] that as $x \to \infty$,

$$\Gamma(1+x) = x^{x+\frac{1}{2}}\, \mathrm{e}^{-x}\, \sqrt{2\pi} \exp\left(\frac{1}{12x} + o\left(\frac{1}{x^2}\right)\right).$$

Since $d_n \sim abn^2/2$ we get that for any fixed $r > 0$ and $p \geq 1$,

$$\left(\frac{\Gamma\left(1+\frac{d_n+r}{p}\right)}{\Gamma\left(1+\frac{d_n}{p}\right)}\right)^{-\frac{1}{r}} = \left(\frac{d_n}{p}\right)^{-\frac{1}{p}} \cdot \left(1 - \frac{p+r}{abp\, n^2} + o\left(\frac{1}{n^2}\right)\right).$$

Therefore, for any $p \geq 1$ and $q > 0$,

$$\left(\frac{\Gamma\left(1+\frac{d_n+q}{p}\right)}{\Gamma\left(1+\frac{d_n}{p}\right)}\right)^{-\frac{1}{q}} \cdot \left(\frac{\Gamma\left(1+\frac{d_n+2}{p}\right)}{\Gamma\left(1+\frac{d_n}{p}\right)}\right)^{\frac{1}{2}} = 1 - \frac{q-2}{abp\, n^2} + o\left(\frac{1}{n^2}\right),$$

which finishes the proof of (14). ∎

3. Moment bounds

Recall that $\mathbb{P}_{n,p}(\mathrm{d}\mathbf{x})$ is the probability measure whose density is defined on $\mathbb{R}^n$ by

$$\mathbb{P}_{n,p}(\mathrm{d}\mathbf{x}) := \frac{1}{Z_{n,p}} \cdot f_{a,b,c}(\mathbf{x})\, \mathrm{e}^{-abn\gamma_p \|\mathbf{x}\|_p^p}\, \mathrm{d}\mathbf{x}, \quad \text{where} \quad f_{a,b,c}(\mathbf{x}) := \prod_{1\leq i<j\leq n} \left|x_i^a - x_j^a\right|^b \cdot \prod_{i=1}^n |x_i|^c.$$

In this section we consider a more general case where a is a positive integer, $b > 0$ and $c \geq 0$. Prior to evaluating their asymptotics, we need to establish some uniform bounds on the quantities

$$\left(\mathbb{E}\,\langle L_{n,p}, h_r\rangle\right)^{\frac{1}{r}} = \left(\int_{\mathbb{R}^n} |x_1|^r\, \mathbb{P}_{n,p}(\mathrm{d}\mathbf{x})\right)^{\frac{1}{r}} \tag{15}$$

where $h_r(x) := |x|^r$. The evaluation of (15) in the case $r = 2$ is the crucial part of the proof in [27] while it was also studied for larger values of r in [17]. We prove here a stronger result. Following the method of proof of Theorem 11.1.2 (i) in [34], which corresponds to the case $a = 1$ and $c = 0$, we establish an upper bound of the tail of the first marginal density of $\mathbb{P}_{n,p}(\mathrm{d}\mathbf{x})$ denoted by $p_n(x_1)$ and defined by

$$p_n(x_1) := \frac{1}{Z_{n,p}} \int_{\mathbb{R}^{n-1}} f_{a,b,c}(x_1, \ldots, x_n)\, \mathrm{e}^{-abn\gamma_p \sum_{i=1}^n |x_i|^p}\, \mathrm{d}x_2 \cdots \mathrm{d}x_n.$$

Theorem 3.1 (Moment and tail bounds on the first marginal)**.** *Let $p \geq 1$ and $R > 0$. Then there exist constants $C, C', C'', X_1, c_1, c' > 0$ depending only on a, b, c and on p, R such that, for every $n \geq 1$:*

(i) for every $|x_1| \geq X_1$,

$$p_n(x_1) \leq \mathrm{e}^{-c_1 n |x_1|^p};$$

(ii) for every $r \geq 1$,

$$\left(\int_{\mathbb{R}^n} |x_1|^r \, \mathbb{P}_{n,p}(\mathrm{d}\mathbf{x}) \right)^{\frac{1}{r}} \leq X_1 + C\Big(\frac{r}{n}\Big)^{\frac{1}{p}},$$

and thus

$$\forall 1 \leq r \leq Rn, \quad \left(\int_{\mathbb{R}^n} |x_1|^r \, \mathbb{P}_{n,p}(\mathrm{d}\mathbf{x}) \right)^{\frac{1}{r}} \leq C'';$$

(iii) for every $i = 1, \ldots, n$, *and every* $B \geq X_1$,

$$\mathbb{P}_{n,p}(|x_i| \geq B) \leq C' \, \mathrm{e}^{-c' n B^p}.$$

Proof. The proof of (i) goes into three steps.
1) We first prove that there exists a constant $C_1 > 0$ independent of the dimension such that

$$\int_{\mathbb{R}^n} |x_1|^p \, \mathbb{P}_{n,p}(\mathrm{d}\mathbf{x}) = \int_{\mathbb{R}} |x_1|^p \, p_n(x_1) \mathrm{d}x_1 \leq C_1.$$

The proof of this inequality follows the lines of the proof of Corollary 7(a) in [17]. For $t > 0$, we define

$$g(t) := \int_{\mathbb{R}^n} f_{a,b,c}(\mathbf{x}) \, \mathrm{e}^{-tabn\gamma_p \|\mathbf{x}\|_p^p} \, \mathrm{d}\mathbf{x}.$$

Then $g(1) = Z_{n,p}$. Changing variable by putting $\mathbf{x} \leftarrow t^{-1/p}\mathbf{y}$ and using that $f_{a,b,c}$ is positively homogeneous of degree $d_n - n$, where $d_n := abn(n-1)/2 + (c+1)n$, we get

$$g(t) = t^{-n/p} \int_{\mathbb{R}^n} f_{a,b,c}(t^{-1/p}\mathbf{y}) \, \mathrm{e}^{-abn\gamma_p \|\mathbf{y}\|_p^p} \, \mathrm{d}\mathbf{y} = t^{-d_n/p} g(1).$$

It follows that $g'(1) = -\frac{d_n}{p} g(1)$. On the other hand, differentiating the integral formula which defines $g(t)$, we also get

$$\begin{aligned} g'(1) &= -abn\gamma_p \int_{\mathbb{R}^n} \|\mathbf{x}\|_p^p \, f_{a,b,c}(\mathbf{x}) \, \mathrm{e}^{-abn\gamma_p \|\mathbf{x}\|_p^p} \, \mathrm{d}\mathbf{x} \\ &= -abn^2\gamma_p \int_{\mathbb{R}^n} |x_1|^p \, f_{a,b,c}(\mathbf{x}) \, \mathrm{e}^{-abn\gamma_p \|\mathbf{x}\|_p^p} \, \mathrm{d}\mathbf{x}. \end{aligned}$$

We conclude that

$$\int_{\mathbb{R}^n} |x_1|^p \, \mathbb{P}_{n,p}(\mathrm{d}\mathbf{x}) = \int_{\mathbb{R}} |x_1|^p \, p_n(x_1) \mathrm{d}x_1 = -\frac{g'(1)}{g(1) abn^2 \gamma_p} = \frac{d_n}{abpn^2\gamma_p}.$$

It follows that $\int_{\mathbb{R}^n} |x_1|^p \, \mathbb{P}_{n,p}(\mathrm{d}\mathbf{x}) \to 1/(2p\gamma_p)$ as $n \to \infty$ and therefore is upper bounded by a constant C_1.

2) In the second step we prove that there exist $c_2, X_2 > 0$ independent of the dimension such that, for all $|x_1| \geq X_2$,

$$p_n(x_1) \leq \frac{Z_{n-1,p}}{Z_{n,p}} \, \mathrm{e}^{-c_2 n |x_1|^p}.$$

For $\mathbf{x} \in \mathbb{R}^n$, we denote

$$g_n(\mathbf{x}) := Z_{n,p} \frac{\mathbb{P}_{n,p}(\mathrm{d}\mathbf{x})}{\mathrm{d}\mathbf{x}} = \prod_{1 \leq i < j \leq n} \left| x_i^a - x_j^a \right|^b \cdot \prod_{i=1}^n |x_i|^c \cdot \mathrm{e}^{-tabn\gamma_p \|\mathbf{x}\|_p^p}.$$

For $\tilde{\mathbf{x}} := (x_2, \dots, x_n) \in \mathbb{R}^{n-1}$ and $\mathbf{x} := (x_1, \tilde{\mathbf{x}})$, using that $n\|\mathbf{x}\|_p^p = n|x_1|^p + (n-1)\|\tilde{\mathbf{x}}\|_p^p + \|\tilde{\mathbf{x}}\|_p^p$ we get

$$g_n(\mathbf{x}) = g_n(x_1, \tilde{\mathbf{x}}) = g_{n-1}(\tilde{\mathbf{x}}) \, \mathrm{e}^{-abn\gamma_p |x_1|^p} \, \mathrm{e}^{-ab\gamma_p \|\tilde{\mathbf{x}}\|_p^p} \, |x_1|^c \prod_{i=2}^n |x_i^a - x_1^a|^b.$$

Hence

$$\begin{aligned} &\frac{1}{Z_{n-1,p}} \int_{\mathbb{R}^{n-1}} g_n(x_1, \tilde{\mathbf{x}}) \mathrm{d}\tilde{\mathbf{x}} \\ &\qquad = |x_1|^c \, \mathrm{e}^{-abn\gamma_p |x_1|^p} \int_{\mathbb{R}^{n-1}} \mathrm{e}^{-ab\gamma_p \|\tilde{\mathbf{x}}\|_p^p} \prod_{i=2}^n |x_i^a - x_1^a|^b \, \mathbb{P}_{n-1,p}(\mathrm{d}\tilde{\mathbf{x}}). \end{aligned} \tag{16}$$

Using that for any $x, y \in \mathbb{R}$ one has $|x - y| \leq |x| + |y| \leq (1 + |x|)(1 + |y|)$, and letting $D_2 := \max_{t \in \mathbb{R}} \mathrm{e}^{-ab\gamma_p t^p} (1 + |t|^a)^b$, we have

$$\frac{Z_{n,p}}{Z_{n-1,p}} \, p_n(x_1) = \frac{1}{Z_{n-1,p}} \int_{\mathbb{R}^{n-1}} g_n(x_1, \tilde{\mathbf{x}}) \mathrm{d}\tilde{\mathbf{x}} \leq |x_1|^c \, \mathrm{e}^{-abn\gamma_p |x_1|^p} (1 + |x_1|^a)^{(n-1)b} D_2^{n-1}.$$

We conclude that for $c_2 = ab\gamma_p/2$ there exists $X_2 > 0$ independent of the dimension such that, for $|x_1| \geq X_2$,

$$p_n(x_1) \leq \frac{Z_{n-1,p}}{Z_{n,p}} \, \mathrm{e}^{-c_2 n |x_1|^p}.$$

3) In the third step, we establish that there exists $c_3 > 0$ such that $\frac{Z_{n-1,p}}{Z_{n,p}} \leq \mathrm{e}^{-c_3 n}$. Integrating equation (16) with respect to $x_1 \in \mathbb{R}$ we get

$$\begin{aligned} &\frac{1}{Z_{n-1,p}} \int_{\mathbb{R}^n} g_n(\mathbf{x}) \mathrm{d}\mathbf{x} \\ &\qquad = \int_{\mathbb{R}} |x_1|^c \, \mathrm{e}^{-abn\gamma_p |x_1|^p} \int_{\mathbb{R}^{n-1}} \mathrm{e}^{-ab\gamma_p \|\tilde{\mathbf{x}}\|_p^p} \prod_{i=2}^n |x_i^a - x_1^a|^b \, \mathbb{P}_{n-1,p}(\mathrm{d}\tilde{\mathbf{x}}) \mathrm{d}x_1. \end{aligned} \tag{17}$$

Applying Jensen's inequality to the integral on $\mathbb{R}^{n-1}$ with the probability measure $\mathbb{P}_{n-1,p}$ and the convex function being the exponential, and applying also the bound obtained in Step 1, we have

$$\int \mathrm{e}^{-ab\gamma_p\|\tilde{\mathbf{x}}\|_p^p}\prod_{i=2}^n|x_i^a-x_1^a|^b\,\mathbb{P}_{n-1,p}(\mathrm{d}\tilde{\mathbf{x}})$$

$$\geq \exp\int\left(-ab\gamma_p\|\tilde{\mathbf{x}}\|_p^p+b\sum_{i=2}^n\log|x_i^a-x_1^a|\right)\mathbb{P}_{n-1,p}(\mathrm{d}\tilde{\mathbf{x}})$$

$$\geq \exp\left(-(n-1)ab\gamma_pC_1+b\int\sum_{i=2}^n\log|x_i^a-x_1^a|\,\mathbb{P}_{n-1,p}(\mathrm{d}\tilde{\mathbf{x}})\right).$$

Plugging this inequality into Equation (17) and noticing that the left-hand side is equal to $Z_{n,p}/Z_{n-1,p}$ we get

$$\frac{Z_{n,p}}{Z_{n-1,p}}\geq \mathrm{e}^{-(n-1)ab\gamma_pC_1}\int_{-\frac12}^{\frac12}|x_1|^c\,\mathrm{e}^{-abn\gamma_p|x_1|^p}\exp\left(b\int\sum_{i=2}^n\log|x_i^a-x_1^a|\,\mathbb{P}_{n-1,p}(\mathrm{d}\tilde{\mathbf{x}})\right)\mathrm{d}x_1.$$

Applying again Jensen's inequality but with the uniform probability measure on $[-\frac12,\frac12]$ we deduce that

$$\frac{Z_{n,p}}{Z_{n-1,p}}\geq \mathrm{e}^{-(n-1)ab\gamma_pC_1}\exp\int_{-\frac12}^{\frac12}\left(c\log|x_1|-abn\gamma_p|x_1|^p\right.$$

$$\left.+b\int\sum_{i=2}^n\log|x_i^a-x_1^a|\,\mathbb{P}_{n-1,p}(\mathrm{d}\tilde{\mathbf{x}})\right)\mathrm{d}x_1.$$

We then use Fubini's theorem and thus want to estimate from below the function g_a defined for $a\geq 1$ being an integer and $x\in\mathbb{R}$ by

$$g_a(x):=\int_{-\frac12}^{\frac12}\log|x^a-y^a|\,\mathrm{d}y.$$

Namely we shall prove that $g_a(x)\geq -a(2\log 2+1)$. First notice that for a even the function $(x,y)\mapsto x^a-y^a$ is even in both variables, and for a odd, splitting the integral and changing variable one has

$$g_a(x)=\int_0^{\frac12}\left(\log|x^a-y^a|+\log|x^a+y^a|\right)\mathrm{d}y=\int_0^{\frac12}\log|x^{2a}-y^{2a}|\,\mathrm{d}y=\frac12\,g_{2a}(x).$$

We are thus reduced to proving the lower bound of g_a in the case where a is even. Then the function g_a is even so we may assume that $x\geq 0$ and we have

$$g_a(x)=2\int_0^{\frac12}\log|x^a-y^a|\,\mathrm{d}y.$$

For $x\geq\frac12$ the function g_a is increasing hence $g_a(x)\geq g_a(1/2)$. Moreover for every $x,y>0$ one has $|x^a-y^a|\geq y^{a-1}|x-y|$, thus

$$g_a(x)\geq 2\int_0^{\frac12}\left((a-1)\log y+\log|x-y|\right)\mathrm{d}y.$$

Let φ be the convex function defined by $\varphi(x) = x\log x$ for $x \geq 0$. A simple calculation shows that, for every $0 \leq x \leq \frac{1}{2}$,

$$g_a(x) \geq -(a-1)(\log 2 + 1) + 2\varphi(x) + 2\varphi\left(\frac{1}{2} - x\right) - 1.$$

From the convexity of φ, one has

$$\varphi(x) + \varphi\left(\frac{1}{2} - x\right) \geq 2\varphi\left(\frac{x + \frac{1}{2} - x}{2}\right) = 2\varphi\left(\frac{1}{4}\right) = -\log 2.$$

We conclude that $g_a(x) \geq -a(2\log 2 + 1)$. Applying this inequality in the lower bound of $Z_{n,p}/Z_{n-1,p}$ we deduce that

$$\begin{aligned}\frac{Z_{n,p}}{Z_{n-1,p}} &\geq \exp\left(-(n-1)ab\gamma_p C_1 - (2\log 2 + 1)\big(c + (n-1)ab\big) - \frac{nab\gamma_p}{(p+1)2^p}\right) \\ &\geq \mathrm{e}^{-c_3 n},\end{aligned}$$

where $c_3 := ab\gamma_p C_1 + (2\log 2 + 1)(c + ab)) + \frac{ab\gamma_p}{(p+1)2^p}$, which finishes our third step. The final conclusion follows from combining the steps 2 and 3 which give that for every $|x_1| \geq X_2$, one has

$$p_n(x_1) \leq \frac{Z_{n-1,p}}{Z_{n,p}}\,\mathrm{e}^{-c_2 n|x_1|^p} \leq \mathrm{e}^{-c_3 n - c_2 n|x_1|^p}.$$

Hence there exist $c_1, X_1 > 0$ such that $p_n(x_1) \leq \mathrm{e}^{-c_1 n|x_1|^p}$ for all $|x_1| \geq X_1$.

(ii) The upper bound for the moment using the bound for the tail is standard and runs as follows. We first cut the integral into two parts and use the bound proved in (i):

$$\begin{aligned}\int_{\mathbb{R}^n} |x_1|^r\,\mathbb{P}_{n,p}(\mathrm{d}\mathbf{x}) = \int_{\mathbb{R}} |t|^r\,p_n(t)\,\mathrm{d}t &\leq X_1^r \int_{|t|\leq X_1} p_n(t)\mathrm{d}t + \int_{|t|\geq X_1} |t|^r\,\mathrm{e}^{-c_1 n|t|^p}\,\mathrm{d}t \\ &\leq X_1^r + 2\int_0^\infty t^r\,\mathrm{e}^{-c_1 n t^p}\,\mathrm{d}t.\end{aligned}$$

Changing variable, this last integral may be written as follows:

$$\int_0^\infty t^r\,\mathrm{e}^{-c_1 n t^p}\,\mathrm{d}t = \frac{1}{p(c_1 n)^{\frac{r+1}{p}}}\int_0^\infty s^{\frac{r+1}{p}-1}\,\mathrm{e}^{-s}\,\mathrm{d}s = \frac{1}{p(c_1 n)^{\frac{r+1}{p}}}\,\Gamma\left(\frac{r+1}{p}\right).$$

Using a standard bound on the Gamma function we conclude that there exists $C > 0$ such that

$$\left(\int_{\mathbb{R}^n} |x_1|^r\;\mathbb{P}_{n,p}(\mathrm{d}\mathbf{x})\right)^{\frac{1}{r}} \leq X_1 + \left(\frac{2}{p(c_1 n)^{\frac{r+1}{p}}}\,\Gamma\left(\frac{r+1}{p}\right)\right)^{\frac{1}{r}} \leq X_1 + C\Big(\frac{r}{n}\Big)^{\frac{1}{p}}.$$

(iii) Using the bound (i), for $B > X_1$ and choosing $c' := c_1/2$, we have

$$\mathbb{P}_{n,p}(|x_i| \geq B) \leq 2\int_B^\infty \mathrm{e}^{-c_1 n t^p}\,\mathrm{d}t \leq 2\,\mathrm{e}^{-c' n B^p}\int_0^\infty \mathrm{e}^{-c' n t^p}\,\mathrm{d}t \leq C'\,\mathrm{e}^{-c' n B^p}.$$

∎

Because of the above uniform bounds, the random measure $L_{n,p}$ is mostly concentrated on a compact interval. As a result, testing $L_{n,p}$ against a general function f is on average not very different from using a truncated version of f instead.

Corollary 3.2 (Truncation argument)**.** *Choose $X_1 \geq 1$ as in Theorem 3.1. Let $B \geq X_1$ and let $\phi\colon \mathbb{R} \to [0,1]$ be a smooth, compactly supported function such that $\phi \equiv 1$ on $[-B, B]$. Then for all continuous, polynomially bounded functions $f, g\colon \mathbb{R} \to \mathbb{R}$, one has*

$$\mathbb{E}\left|g\big(\langle L_{n,p}, f\rangle\big) - g\big(\langle L_{n,p}, f\phi\rangle\big)\right| = O(\alpha^n),$$

where $\alpha := \mathrm{e}^{-\frac{c' B^p}{2}} < 1$.

Proof. As f and g are polynomially bounded, there exist constants $A_1, A_2, s, t \geq 1$ such that for every $y \in \mathbb{R}$, $|f(y)| \leq A_1 + |y|^s$ and $|g(y)| \leq A_2 + |y|^t$. Hence, we can find constants $A, r \geq 1$ such that, for every $\mathbf{x} \in \mathbb{R}^n$,

$$|g(\langle L_{n,p}(\mathbf{x}), f\rangle)| = \left|g\left(\frac{1}{n}\sum_{i=1}^n f(x_i)\right)\right| \leq A + \frac{1}{n}\sum_{i=1}^n |x_i|^r = A + \langle L_{n,p}(\mathbf{x}), h_r\rangle$$

(e.g., $A := 2^t A_1 + A_2$ and $r := st$, using Jensen's inequality). Since $0 \leq \phi \leq 1$ on $\mathbb{R}$, we get the same bound for

$$|g(\langle L_{n,p}(\mathbf{x}), f\phi\rangle)| \leq A + \langle L_{n,p}(\mathbf{x}), h_r\rangle.$$

As $\phi \equiv 1$ on $[-B, B]$,

$$|g(\langle L_{n,p}(\mathbf{x}), f\rangle) - g(\langle L_{n,p}(\mathbf{x}), f\phi\rangle)| \leq 2\mathbb{1}_{\{\exists i : |x_i| > B\}}\big(A + \langle L_{n,p}(\mathbf{x}), h_r\rangle\big).$$

Since r is a constant, we have by Theorem 3.1.(ii),

$$\mathbb{E}\,\langle L_{n,p}(\mathbf{x}), h_r\rangle^2 = \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^n |x_i|^r\right)^2 \leq \mathbb{E}\,\frac{1}{n}\sum_{i=1}^n |x_i|^{2r} \leq C''^{2r}.$$

Combining Theorem 3.1.(iii) with a union bound,

$$\mathbb{P}_{n,p}(\exists i : |x_i| > B) \leq nC'\,\mathrm{e}^{-c' n B^p}.$$

Therefore, by Cauchy-Schwarz inequality, we conclude that

$$\mathbb{E}\left|g\big(\langle L_{n,p}, f\rangle\big) - g\big(\langle L_{n,p}, f\phi\rangle\big)\right| \leq 2nC'\,\mathrm{e}^{-c' n B^p}(A + C''^r).$$

This is $O(\alpha^n)$ with e.g. $\alpha = \mathrm{e}^{-\frac{c' B^p}{2}}$. ∎

4. Asymptotics

In this section, we prove Theorems 1.1 and 1.2 by completing the asymptotic expansions initiated in Lemma 2.2. As previously explained, we will appeal to the literature on random matrices, as $L_{n,p}$ corresponds to the empirical distribution of the so called β-ensemble with potential $V(x) \propto |x|^p$ and $Z_{n,p}$ is the so called partition function. We split the discussion into two cases, according to Table 1. In the first case ($a = 1$), we

prove both Theorems 1.1 and 1.2, where matrices in $B_E(S_p^n)$ all have symmetries with respect to their diagonal and $L_{n,p}$ thus corresponds to empirical distributions of real eigenvalues. The second case ($a = 2$) pertains to Theorem 1.1 only and is treated more conveniently by working with $\mathbb{R}^+$-valued measures.

4.1. **The self-adjoint case ($a = 1$).** In view of Table 1, we have $a = 1$, $b = \dim_{\mathbb{R}}(\mathbb{F})$ and $c = 0$. Hence, from (12), the probability measure $\mathbb{P}_{n,p}$ can be written as a Gibbs measure

$$\mathbb{P}_{n,p}(\mathrm{d}\mathbf{x}) = \frac{1}{Z_{n,p}} \cdot \mathrm{e}^{-\frac{b}{2}n^2 H_{n,p}(\mathbf{x})}\, \mathrm{d}\mathbf{x}, \tag{18}$$

with the Hamiltonian

$$H_{n,p}(\mathbf{x}) := \frac{2}{n}\sum_{i=1}^{n} \gamma_p |x_i|^p - \frac{1}{n^2}\sum_{i\neq j} \log|x_i - x_j|,$$

where γ_p is defined in (8). Heuristically this Hamiltonian is approximated as $n \to \infty$ by

$$\mathcal{I}_p(\mu) := 2\int_{\mathbb{R}} \gamma_p |x|^p \,\mu(\mathrm{d}x) - \iint_{\mathbb{R}^2_{\neq}} \log|x-y|\,\mu(\mathrm{d}x)\mu(\mathrm{d}y),$$

where $\mathbb{R}^2_{\neq} := \{(x,y) \in \mathbb{R}^2 \colon x \neq y\}$, provided that the probability measure μ is sufficiently close to the empirical measure $L_{n,p}(\mathbf{x}) := n^{-1}\sum_{i=1}^n \delta_{x_i}$. Thus, because of (18), we expect that $L_{n,p}$ concentrates as $n \to \infty$ around a probability measure minimizing the functional $\mathcal{I}_p$. This idea constitutes the cornerstone of large deviation principles for the empirical spectral distribution of large random matrices, as originally derived by Ben Arous and Guionnet [9]. The next two lemmas formalize what we need. We refer to the textbooks [14, Chapter 6] and [2, Section 2.6] or the recent article [15] for deeper results.

Lemma 4.1 (Equilibrium measure)**.** *There exists a unique element μ_p minimizing the functional $\mathcal{I}_p$ over the space $\mathcal{P}(\mathbb{R})$ of real probability measures:*

$$\inf_{\mu\in\mathcal{P}(\mathbb{R})} \mathcal{I}_p(\mu) = \mathcal{I}_p(\mu_p).$$

Furthermore, μ_p has the following properties:

(i) μ_p is compactly supported, with support $[-1,1]$;
(ii) μ_p is absolutely continuous with respect to the Lebesgue measure, with density

$$f_p(x) := \frac{p|x|^{p-1}}{\pi} \int_{|x|}^1 \frac{v^{-p}}{\sqrt{1-v^2}}\,\mathrm{d}v, \qquad |x| \leq 1;$$

(iii) $\mathcal{I}_p(\mu_p) = \log 2 + \frac{3}{2p}$.

Proof. The existence and uniqueness of the minimizer μ_p follows from [36, Theorem I.1.3] with the weight function $w\colon x \mapsto \exp(-\gamma_p|x|^p)$ on $\mathbb{R}$. Properties (i)–(iii) are provided by [36, Theorem IV.5.1]. We have rewritten the density with the change of variable $v \leftarrow |x|/u$. ∎

Lemma 4.2 (Convergence towards the equilibrium measure)**.** *Let $L_{n,p}$ be the empirical measure of $\mathbf{x}$ drawn from the probability $\mathbb{P}_{n,p}$ defined in (18). Then:*

(i) For every continuous, polynomially bounded functions $f, g\colon \mathbb{R} \to \mathbb{R}$, it holds that

$$\lim_{n\to\infty} \mathbb{E}\, g(\langle L_{n,p}, f\rangle) = g(\langle \mu_p, f\rangle).$$

(ii) We have

$$\lim_{n\to\infty} -\frac{1}{d_n} \log Z_{n,p} = \log 2 + \frac{3}{2p},$$

where the normalizing constant $Z_{n,p}$ is defined in (18).

Proof. (i) By Corollary 3.2, we may reduce to f bounded. Thus the random variable $g(\langle L_{n,p}, f\rangle)$ is uniformly bounded, and letting $\Delta_{n,p}(f) := \langle L_{n,p}, f\rangle - \langle \mu_p, f\rangle$ we see from the Borel-Cantelli lemma and the dominated convergence theorem that it suffices to show that

$$\sum_{n=1}^{\infty} \mathbb{P}(|\Delta_{n,p}(f)| > \varepsilon_n) < \infty,$$

for some null sequence (ε_n). Thanks to [19, Theorem 2.1], we have that

$$\alpha_n := \frac{1}{n} \log \mathbb{E} \exp\big(n|\Delta_{n,p}(f)|\big) \xrightarrow[n\to\infty]{} 0.$$

Choosing now ε_n as, e.g., $\varepsilon_n := \alpha_n/2 + (\log n)/n$, we see from Markov's inequality that

$$\mathbb{P}(|\Delta_{n,p}(f)| > 2\varepsilon_n) = \mathbb{P}(\exp(n|\Delta_{n,p}(f)|) > \exp(2n\varepsilon_n)) = O\big(e^{-(2\varepsilon_n - \alpha_n)n}\big) = O\left(\frac{1}{n^2}\right),$$

and the first point of the lemma is proved.

(ii) This point follows from [19, Corollary 4.3] and Lemma 4.1.(iii). ∎

We now have all the ingredients to give a simple proof of Theorem 1.1 in the case $a = 1$.

Proof of Theorem 1.1, case $a = 1$. We start by recalling (13): as $n \to \infty$,

$$\frac{1}{\sqrt{d_n}}\, I_q(B_E(S_p^n)) \sim \frac{\mathrm{e}^{-\frac{1}{p}-\frac{3}{4}}}{\sqrt{2\pi}}\, Z_{n,p}^{-1/d_n} \left(\mathbb{E}\,\langle L_{n,p}, h_2\rangle^{\frac{q}{2}}\right)^{\frac{1}{q}}.$$

By Lemma 4.2, we have $Z_{n,p}^{-1/d_n} \to 2\,\mathrm{e}^{\frac{3}{2p}}$ and

$$\lim_{n\to\infty} \mathbb{E}\,\langle L_{n,p}, h_2\phi\rangle^{\frac{q}{2}} = \langle \mu_p, h_2\phi\rangle^{\frac{q}{2}}.$$

Plugging in the value of $\langle \mu_p, h_2\rangle$ computed in Lemma 5.1, we conclude that

$$\lim_{n\to\infty} \frac{1}{\sqrt{d_n}}\, I_q(B_E(S_p^n)) = \mathrm{e}^{\frac{1}{2p}-\frac{3}{4}} \sqrt{\frac{p}{\pi(p+2)}}.$$ ∎

The proof of Lemma 4.2 informs us rather poorly on the speed of convergence of the linear statistics $\langle L_{n,p}, f\rangle$ towards $\langle \mu_p, f\rangle$. In fact, a necessary condition for Theorem 1.2 to hold is

$$\operatorname{Var}\langle L_{n,p}, h_2\rangle = O\Big(\frac{1}{n^2}\Big),$$

so we expect that $\langle L_{n,p}, h_2\rangle - \langle \mu_p, h_2\rangle = O(n^{-1})$. Besides, to identify the actual limit in Theorem 1.2, we need to understand precisely the asymptotic behavior of the fluctuations $n(\langle L_{n,p}, h_2\rangle - \langle \mu_p, h_2\rangle)$.

There has been an increasing wealth of literature on such fluctuations of linear statistics. Given a general potential V, let $L_n := n^{-1}\sum_{i=1}^n \delta_{x_i}$ be the empirical distribution associated with an ensemble $\mathbf{x} := (x_1, \dots, x_n) \in \mathbb{R}^n$ of particles which are subject to the confining potential V and pairwise repulsive logarithmic interaction. Then, the convergence to a Gaussian distribution as $n \to \infty$ of the random variable

$$F_n(\xi) := n\Big(\langle L_n, \xi\rangle - \langle \mu_V, \xi\rangle\Big), \tag{19}$$

where ξ is a given test function and μ_V is the equilibrium distribution, has been widely studied. The regularity of the external potential V plays a prominent role. It was assumed to be a polynomial of even degree with positive leading coefficient in the seminal paper of Johansson [19]. This condition has then been relaxed during the last two decades to include real-analytic potentials [11, 28, 38] and, more recently, potentials of class $\mathcal{C}^r$ with r reasonably large [7, 8, 29].

In our setting (specifically, of Theorem 1.2), the potential is $V = V_p := 2\gamma_p h_p$ with a lack of regularity at 0, the β-ensemble $\mathbf{x}$ has the distribution $\mathbb{P}_{n,p}$ with $a = 1$, $b = \beta$, and $c = 0$ (cf. (18)), and the equilibrium measure μ_V is of course the probability distribution μ_p of Lemmas 4.1 and 4.2. We only need to establish the central limit theorem for $\xi = h_2$, that is the convergence of $F_n(h_2)$. As V_p is not always a polynomial nor real-analytic, we choose to work with the currently most general version due to Bekerman, Leblé and Serfaty [8]. Like in Johansson [19], the central limit theorem is obtained by establishing convergence of the moment generating function (a.k.a. Laplace transform) of (19).

Proposition 4.3 (Fluctuations of the linear statistics)**.** *Let $p \in (3, \infty)$ and let*

$$F_{n,p} := n\Big(\langle L_{n,p}, h_2\phi\rangle - \langle \mu_p, h_2\rangle\Big),$$

with ϕ being the truncation function of Corollary 3.2. Then $(F_{n,p}^2)_{n\geq 1}$ is uniformly integrable, i.e.,

$$\lim_{K\to\infty} \limsup_{n\to\infty} \mathbb{E}\, F_{n,p}^2 \mathbb{1}_{\{F_{n,p}^2 > K\}} = 0.$$

Furthermore, $\lim_{n\to\infty} \operatorname{Var} F_{n,p} = \frac{1}{4b}$.

Remark 4.4. *When $p \geq 6$, this proposition is a direct consequence of [8, Theorem 1]. Indeed, we are in the so called "one cut" regime which corresponds to $\mathsf{m} = \mathsf{n} = \mathsf{k} = 0$, and we remark that $\xi = h_2\phi$ is $\mathcal{C}^\infty$ and $V = V_p$ is $\mathcal{C}^6$ when $p \geq 6$.*

Observe that for $p \geq 8$, this is also a direct consequence of the CLT of Lambert, Ledoux, and Webb [29, Theorem 1.2], which they derived alternatively using Stein's method.

Proof. To explain why it is enough to assume $p > 3$, and for the sake of clarity, we reproduce the key arguments of Bekerman, Leblé and Serfaty [8], introducing in particular the key master equation (21), and underlining where the regularity of its solution is required.

The goal is to prove that, when $p \in (3, \infty)$, the moment generating function of $F_{n,p}$ converges to that of a Gaussian variable N with variance $1/4b$, that is, there exists $m_p := m_p(h_2\phi) \in \mathbb{R}$ such that

$$\lim_{n\to\infty} \mathbb{E}\, \mathrm{e}^{sF_{n,p}} = \exp\left(s m_p + \frac{s^2}{2}\left(\frac{1}{4b}\right)\right) \tag{20}$$

holds for all $s \in \mathbb{R}$ with $|s|$ sufficiently small. It entails the convergence in distribution of $F_{n,p}$ towards $N \sim \mathcal{N}(m_p, \frac{1}{4b})$, together with the convergence of all moments $\mathbb{E}\, F_{n,p}^k \to \mathbb{E}\, N^k$, $k \in \mathbb{N}$. In particular, $(F_{n,p}^2)_{n\geq 1}$ will be uniformly integrable and $\lim_{n\to+\infty} \operatorname{Var} F_{n,p} = 1/4b$.

First, by Theorem 3.1.(iii) and the union bound, we can choose $B > 1$ sufficiently large such that $\mathbb{P}_{n,p}(\exists i \leq n : |x_i| \geq B) \leq C' n\, \mathrm{e}^{-c' n B^p}$ for some constants $c', C' > 0$. Observing that $\|F_{n,p}(h_2\phi)\|_\infty = O(n)$ we deduce that as soon as $|s|$ is sufficiently small,

$$\lim_{n\to\infty} \mathbb{E}\, \mathrm{e}^{sF_{n,p}} \mathbb{1}_{\{\exists i\leq n : |x_i|\geq B\}} = 0.$$

Thus, we may restrict to $\mathbf{x} \in U_0^n$ with $U_0 := (-B, B)$. Now, the strategy, adopted in [8] and developed already in [19], is to perturb the Hamiltonian $H_{n,p}$ in (18) as follows:

$$H_{n,p}^t(\mathbf{x}) := H_{n,p}(\mathbf{x}) + \frac{t}{n}\sum_{i=1}^n x_i^2 = \frac{1}{n}\sum_{i=1}^n \big(V_p(x_i) + t x_i^2\big) - \frac{1}{n^2}\sum_{1\leq i\neq j\leq n} \log|x_i - x_j|,$$

where we have set $t := -\frac{2s}{bn}$. Then, we write

$$\mathbb{E}\, \mathrm{e}^{sF_{n,p}} = o(1) + \mathrm{e}^{-sn\langle \mu_p, h_2\rangle}\, \mathbb{E}\, \mathrm{e}^{sn\langle L_{n,p}, h_2\rangle} \mathbb{1}_{U_0^n} = o(1) + \frac{\mathrm{e}^{-sn\langle \mu_p, h_2\rangle}}{Z_{n,p}} \int_{U_0^n} \mathrm{e}^{-\frac{b}{2}n^2 H_{n,p}^t(\mathbf{x})}\, \mathrm{d}\mathbf{x}.$$

Next, one applies in the last integral a change of variables $x_i \leftarrow \vartheta_t(y_i)$ for $1 \leq i \leq n$ and for some well-chosen $\mathcal{C}^1$-diffeomorphism $\vartheta_t \colon U_t \to U_0$ depending on t. We get

$$\begin{aligned}\mathbb{E}\, \mathrm{e}^{sF_{n,p}} &= o(1) + \frac{\mathrm{e}^{-sn\langle \mu_p, h_2\rangle}}{Z_{n,p}} \int_{U_t^n} \mathrm{e}^{-\frac{b}{2}n^2 H_{n,p}^t\circ\vartheta_t(\mathbf{y})} \prod_{i=1}^n |\vartheta_t'(y_i)|\, \mathrm{d}\mathbf{y} \\ &= o(1) + \mathrm{e}^{-sn\langle \mu_p, h_2\rangle}\, \mathbb{E}\, \mathrm{e}^{-\frac{b}{2}n^2 (H_{n,p}^t\circ\vartheta_t - H_{n,p}) + n\langle L_{n,p}, \log|\vartheta_t'|\rangle} \mathbb{1}_{U_t^n}.\end{aligned}$$

The idea is that, for a judicious choice of ϑ_t, the exponent in the last expectation becomes simple enough to establish the convergence towards the moment generating function of a Gaussian distribution. The diffeomorphism is chosen as $\vartheta_t(y) := y + t\psi_p(y)$ where, in our setting, $\psi_p \colon \mathbb{R} \to \mathbb{R}$ is the solution to the equation

$$\Xi_p \psi = \frac{1}{2} h_2 + c_{h_2} \tag{21}$$

for some constant c_{h_2}, where Ξ_p is the so called "master operator" acting on $\mathcal{C}^1$ functions and defined for $x \in \mathbb{R}$ by

$$\Xi_p \psi(x) = -\frac{1}{2} V_p'(x) \psi(x) + \int \frac{\psi(x) - \psi(y)}{x - y} \, \mu_p(\mathrm{d}y).$$

We show in Section 5.2 that ψ_p is an odd function, we give its explicit expression and check that it is of class $\mathcal{C}^{\lceil p \rceil - 1} \subset \mathcal{C}^1$. Then, by the local inversion theorem, $\vartheta_t = \mathrm{Id} + t\psi_p$ becomes a $\mathcal{C}^1$-diffeomorphism $U_t \to U_0$ provided $|t| = 2|s|/bn \leq \tau$ is sufficiently small (equivalently, n is sufficiently large), and we have

$$\mathbb{E}\, \mathrm{e}^{sF_{n,p}} = o(1) + \mathrm{e}^{-sn\langle \mu_p, h_2 \rangle} \, \mathbb{E}\, \mathrm{e}^{-\frac{b}{2} n^2 \left(H^t_{n,p} \circ \vartheta_t - H_{n,p} \right) + n \langle L_{n,p}, \log(1 + t\psi_p') \rangle} \mathbb{1}_{U_t^n}, \tag{22}$$

where $U_t := (-A_t, A_t)$ and $\vartheta_t(A_t) = A_t + t\psi_p(A_t) = B$. Furthermore, we know from the implicit function theorem that A_t, $|t| \leq \tau$, depends continuously on t, and thus choosing τ small enough we can assume that $1 < \inf_{|t| \leq \tau} A_t < \sup_{|t| \leq \tau} A_t < \infty$.

The remaining of the proof is to Taylor-expand as $t \to 0$ (i.e., $n \to \infty$) the terms $H^t_{n,p} \circ \vartheta_t - H_{n,p}$ and $\log(1 + t\psi_p')$ appearing in the right-hand side of (22) up to the order $O(t^3)$. After a rather lengthy but not difficult calculation (see [8, Section 4]), and using that ψ_p solves (21), we obtain that

$$\mathbb{E}\, \mathrm{e}^{sF_{n,p}} = o(1) + \mathrm{e}^{sm_p + \frac{s^2}{2} \Sigma^2(h_2)} \, \mathbb{E} \exp\left\{ -\frac{s}{n} \mathsf{A}_n[\psi_p] + O(nt^2) + O(n^2 t^3) \right\} \mathbb{1}_{U_t^n}, \tag{23}$$

where $O(nt^2)$ and $O(n^2t^3)$ are random quantities converging uniformly to 0 as $n \to \infty$ (recall that $t := -\frac{2s}{bn}$), $m_p := \langle \mu_p, \psi_p' \rangle$ is the limiting mean,

$$\Sigma^2(h_2) := \frac{1}{2b\pi^2} \int_{-1}^{1} \int_{-1}^{1} \frac{(x+y)^2(1-xy)}{\sqrt{1-x^2}\sqrt{1-y^2}} \, \mathrm{d}x \mathrm{d}y$$

is the limiting variance, and, lastly,

$$\mathsf{A}_n[\psi_p] := n^2 \iint \frac{\psi_p(x) - \psi_p(y)}{x - y} (L_{n,p} - \mu_p)(\mathrm{d}x)(L_{n,p} - \mu_p)(\mathrm{d}y)$$

is the so called *anisotropy term*. A careful inspection of the proof of [8, Proposition 5.4] shows that, for $|s|$ sufficiently small,

$$\lim_{n \to \infty} \log \mathbb{E} \exp\left\{ -\frac{s}{n} \mathsf{A}_n[\psi_p] \right\} = 0 \tag{24}$$

holds provided ψ_p is of class $\mathcal{C}^3$. This is true as soon as $p > 3$; we postpone the proof of this key technical point to Section 5.2, see Lemma 5.3. Note that (24) is the only place where the hypothesis $p > 3$ is used. It then follows by the Cauchy-Schwarz inequality that the expectation in the right-hand side of (23) tends to 1. As the value of $\Sigma^2(h_2)$ simplifies to $1/4b$ thanks to Lemma 5.2, this establishes (20). ∎

We can now give a proof of Theorem 1.2.

Proof of Theorem 1.2. Recall the earlier computation (14) (with $a = 1$): as $n \to \infty$,

$$\frac{I_q(B_E(S_p^n))}{I_2(B_E(S_p^n))} = \left(1 - \frac{q-2}{bpn^2} + o\left(\frac{1}{n^2}\right)\right) \frac{\left(\mathbb{E}\,\langle L_{n,p}, h_2\rangle^{q/2}\right)^{1/q}}{\left(\mathbb{E}\,\langle L_{n,p}, h_2\rangle\right)^{1/2}}. \tag{25}$$

We must expand both the numerator and denominator of that latter fraction. By Corollary 3.2, referring to the truncation function ϕ there, we may replace h_2 by $h_2\phi$ as this only induces an $o(n^{-2})$ error:

$$\frac{\mathbb{E}\,\langle L_{n,p}, h_2\rangle^{q/2}}{\left(\mathbb{E}\,\langle L_{n,p}, h_2\rangle\right)^{q/2}} = \frac{\mathbb{E}\,\langle L_{n,p}, h_2\phi\rangle^{q/2} + o(\frac{1}{n^2})}{\left(\mathbb{E}\,\langle L_{n,p}, h_2\phi\rangle + o(\frac{1}{n^2})\right)^{q/2}}. \tag{26}$$

Applying Taylor-Lagrange's formula to the function $x \mapsto x^{q/2}$ between $x_0 := \langle \mu_p, h_2\rangle > 0$ and $x_0 + h$ yields

$$\left|(x_0+h)^{\frac{q}{2}} - x_0^{q/2} - \frac{q}{2}hx_0^{q/2-1} - \frac{q(q-2)}{8}h^2x_0^{q/2-2}\right| \mathbb{1}_{\{|h|\le \frac{x_0}{2}\}} \le C_{q,x_0}\,|h|^3,$$

for some constant C_{q,x_0} depending on q and x_0. Up to enlarging C_{q,x_0}, we also have

$$\left|x_0^{q/2} + \frac{q}{2}hx_0^{q/2-1} + \frac{q(q-2)}{8}h^2x_0^{q/2-2}\right| \mathbb{1}_{\{|h|> \frac{x_0}{2}\}} \le C_{q,x_0}\,h^2 \mathbb{1}_{\{|h|> \frac{x_0}{2}\}}.$$

We write $\langle L_{n,p}, h_2\phi\rangle = x_0 + h$ with $h := \frac{1}{n}F_{n,p} = \langle L_{n,p}, h_2\phi\rangle - \langle \mu_p, h_2\rangle$ and note that $|h| \le \|h_2\phi\|_\infty + x_0$. Then, on the one hand, using the above inequalities according to whether $|h| \le x_0/2$ or $|h| > x_0/2$ gives

$$\begin{aligned}
n^2 &\left|\mathbb{E}\,\langle L_{n,p}, h_2\phi\rangle^{\frac{q}{2}} \mathbb{1}_{\{|F_{n,p}|\le \frac{nx_0}{2}\}} - x_0^{q/2} - \frac{qx_0^{q/2-1}}{2n}\,\mathbb{E}\,F_{n,p} - \frac{q(q-2)x_0^{q/2-2}}{8n^2}\,\mathbb{E}\,F_{n,p}^2\right| \\
&\le C_{q,x_0}\left(\frac{\mathbb{E}\,|F_{n,p}|^3}{n} + \mathbb{E}\,F_{n,p}^2 \mathbb{1}_{\{|F_{n,p}|> \frac{nx_0}{2}\}}\right) \\
&\le C_{q,x_0}\left(\frac{K^{3/2}}{n} + (\|h_2\phi\|_\infty + x_0)\,\mathbb{E}\,F_{n,p}^2 \mathbb{1}_{\{F_{n,p}^2>K\}} + \mathbb{E}\,F_{n,p}^2 \mathbb{1}_{\{|F_{n,p}|> \frac{nx_0}{2}\}}\right) \\
&\le C_{q,x_0}\left(\frac{K^{3/2}}{n} + (\|h_2\phi\|_\infty + x_0 + 1)\,\mathbb{E}\,F_{n,p}^2 \mathbb{1}_{\{F_{n,p}^2>K\}}\right), \qquad (27)
\end{aligned}$$

for any $0 < K \le n^2x_0^2/4$. On the other hand, for such n and K,

$$n^2\,\mathbb{E}\,\langle L_{n,p}, h_2\phi\rangle^{\frac{q}{2}} \mathbb{1}_{\{|F_{n,p}|> \frac{nx_0}{2}\}} \le \frac{4}{x_0^2}\|h_2\phi\|_\infty^{q/2}\,\mathbb{E}\,F_{n,p}^2 \mathbb{1}_{\{F_{n,p}^2>K\}}. \tag{28}$$

Proposition 4.3 tells us that the right-hand sides of (27) and (28) vanish (letting $n \to \infty$ first then $K \to \infty$), and that $\operatorname{Var} F_{n,p} \to \frac{1}{4b}$ as $n \to \infty$. Hence, by triangle inequality

$$\mathbb{E}\,\langle L_{n,p}, h_2\phi\rangle^{\frac{q}{2}} = \langle \mu_p, h_2\rangle^{\frac{q}{2}}\left(1 + \frac{q\,\mathbb{E}\,F_{n,p}}{2n\langle \mu_p, h_2\rangle} + \frac{q(q-2)\,\mathbb{E}\,F_{n,p}^2}{8n^2\langle \mu_p, h_2\rangle^2} + o\left(\frac{1}{n^2}\right)\right).$$

This holds in particular when $q = 2$, and going back to (26) we get

$$\begin{aligned}\frac{\mathbb{E}\,\langle L_{n,p}, h_2\rangle^{q/2}}{\left(\mathbb{E}\,\langle L_{n,p}, h_2\rangle\right)^{q/2}} &= \frac{\mathbb{E}\,\langle L_{n,p}, h_2\phi\rangle^{q/2} + o(\frac{1}{n^2})}{\left(\mathbb{E}\,\langle L_{n,p}, h_2\phi\rangle + o(\frac{1}{n^2})\right)^{q/2}} \\ &= \frac{1 + \frac{q\,\mathbb{E}\,F_{n,p}}{2n\langle\mu_p,h_2\rangle} + \frac{q(q-2)\,\mathbb{E}\,F_{n,p}^2}{8n^2\langle\mu_p,h_2\rangle^2} + o\big(\frac{1}{n^2}\big)}{1 + \frac{q\,\mathbb{E}\,F_{n,p}}{2n\langle\mu_p,h_2\rangle} + \frac{q(q-2)\,(\mathbb{E}\,F_{n,p})^2}{8n^2\langle\mu_p,h_2\rangle^2} + o\big(\frac{1}{n^2}\big)} \\ &= 1 + \frac{q(q-2)}{8\langle\mu_p,h_2\rangle^2}\cdot\frac{\operatorname{Var} F_{n,p}}{n^2} + o\left(\frac{1}{n^2}\right) \\ &= 1 + \frac{q(q-2)(p+2)^2}{8bp^2\,n^2} + o\left(\frac{1}{n^2}\right),\end{aligned}$$

where we also replaced the value of $\langle\mu_p, h_2\rangle$ computed in Lemma 5.1. Raising this expansion to the power $1/q$ and returning to (25), we obtain

$$\frac{I_q(B_E(S_p^n))}{I_2(B_E(S_p^n))} = 1 + \frac{(q-2)(p-2)^2}{16p^2\,d_n} + o\left(\frac{1}{d_n}\right),$$

because $d_n \sim bn^2/2$ (for $a = 1$). ∎

4.2. **The case $E = \mathcal{M}_n(\mathbb{F})$ ($a = 2$).** We suppose in this section that $a = 2$. We reduce to measures and integrals over $\mathbb{R}^+$ by performing the change of variables $y_i := |x_i|^2$ for all $1 \le i \le n$. Specifically, the pushforward of $\mathbb{P}_{n,p}$ by the map $\mathbf{x} \in \mathbb{R}^n \mapsto \mathbf{y} \in (\mathbb{R}^+)^n$ is the probability measure[1] with density

$$\widetilde{\mathbb{P}}_{n,p}(\mathrm{d}\mathbf{y}) := \frac{1}{Z_{n,p}}\cdot\prod_{1\le i<j\le n}|y_i - y_j|^b\cdot\prod_{i=1}^n y_i^{\frac{c-1}{2}}\cdot \mathrm{e}^{-2bn\gamma_p\|\mathbf{y}\|_{p/2}^{p/2}}\,\mathrm{d}\mathbf{y}, \qquad \mathbf{y}\in(\mathbb{R}^+)^n.$$

In particular, for every measurable functions $f\colon \mathbb{R}^+ \to \mathbb{R}$, and $g\colon \mathbb{R} \to \mathbb{R}$,

$$\int_{(\mathbb{R}^+)^n} g\left(\frac{1}{n}\sum_{i=1}^n f(y_i)\right)\widetilde{\mathbb{P}}_{n,p}(\mathrm{d}\mathbf{y}) \;=\; \int_{\mathbb{R}^n} g\left(\frac{1}{n}\sum_{i=1}^n f\big(x_i^2\big)\right)\mathbb{P}_{n,p}(\mathrm{d}\mathbf{x}). \tag{29}$$

It is here convenient to work with the empirical probability measure $\widetilde{L}_{n,p} := n^{-1}\sum_{i=1}^n \delta_{y_i}$ where $\mathbf{y} \in (\mathbb{R}^+)^n$ is sampled from $\widetilde{\mathbb{P}}_{n,p}$, and we note that Corollary 3.2 obviously still applies with $\widetilde{L}_{n,p}$ in place of $L_{n,p}$. Similarly to the previous section, we introduce

$$\widetilde{\mathcal{I}}_p(\mu) := 4\int_{\mathbb{R}^+}\gamma_p|y|^{p/2}\,\mu(\mathrm{d}y) - \iint_{(\mathbb{R}^+)^2_{\neq}}\log|x-y|\,\mu(\mathrm{d}x)\mu(\mathrm{d}y).$$

The counterparts of Lemmas 4.1 and 4.2 are as follows.

[1]The normalizing constant $Z_{n,p}$ remains unchanged; it is just as in (10) but with $a = 2$.

Lemma 4.5 (Equilibrium measure)**.** *There exists a unique element $\widetilde{\mu}_p$ minimizing the functional $\widetilde{\mathcal{I}}_p$ over the space $\mathcal{P}(\mathbb{R}^+)$ of probability measures on $\mathbb{R}^+$:*

$$\widetilde{\mathcal{I}}_p(\widetilde{\mu}_p) \;=\; \inf_{\mu\in\mathcal{P}(\mathbb{R}^+)} \widetilde{\mathcal{I}}_p(\mu).$$

Furthermore, $\widetilde{\mu}_p$ coincides with the image measure of μ_p by the map $x\mapsto x^2$, and it has the following properties:

(i) $\widetilde{\mu}_p$ is compactly supported, with support $[0,1]$;
(ii) $\widetilde{\mu}_p$ is absolutely continuous with respect to the Lebesgue measure, with density

$$\frac{\mathrm{d}\widetilde{\mu}_p}{\mathrm{d}y} = \frac{p y^{\frac{p}{2}-1}}{\pi} \int_{\sqrt{y}}^1 \frac{u^{-p}}{\sqrt{1-u^2}}\,\mathrm{d}u, \qquad 0\le y\le 1;$$

(iii) $\widetilde{\mathcal{I}}_p(\widetilde{\mu}_p) = 2\log 2 + \frac{3}{p}$.

Proof. In the previous section, the minimization over all *real* probability measures of the functional $\mathcal{I}_p$ corresponding to the weight function $w\colon x\mapsto \exp(-\gamma_p|x|^p)$ gave rise to the minimizer μ_p. We are now facing the minimization problem for probability measures on $\mathbb{R}^+ = \{x^2 : x\in\mathbb{R}\}$, with the weight function being $v\colon y\mapsto \exp(-2\gamma_p\, y^{p/2}) = w(\sqrt{y})^2$. According to [36, Theorem IV.1.10.(f)], the solution of the latter is simply the image measure of μ_p by the map $x\mapsto x^2$. From this, (i)–(iii) easily follow. ∎

Lemma 4.6 (Convergence towards the equilibrium measure)**.**

(i) For every continuous, polynomially bounded functions $f, g\colon \mathbb{R}^+\to\mathbb{R}$, it holds that

$$\lim_{n\to\infty} \mathbb{E}\, g\Big(\langle \widetilde{L}_{n,p}, f\rangle\Big) = g(\langle\widetilde{\mu}_p, f\rangle).$$

(ii) We have

$$\lim_{n\to\infty} -\frac{1}{d_n}\log Z_{n,p} = \log 2 + \frac{3}{2p}.$$

Proof. We apply[2] [18, Theorem 5.5.1]. This theorem provides the limit of $Z_{n,p}^{1/d_n}$ stated in (ii), as well as a large deviation principle with good rate function $\frac{b}{2}\big(\widetilde{\mathcal{I}}_p - \widetilde{\mathcal{I}}_p(\widetilde{\mu}_p)\big)$ for the random probability measures $\widetilde{L}_{n,p}$. By the Borel-Cantelli lemma (see e.g. [18, p. 212] for details), this entails that $\widetilde{L}_{n,p}$ converges almost surely towards $\widetilde{\mu}_p$ in the sense of weak convergence of probability measures, that is $\langle\widetilde{L}_{n,p}, f\rangle \to \langle\widetilde{\mu}_p, f\rangle$ $\mathbb{P}$-a.s. for every continuous, bounded function $f\colon\mathbb{R}^+\to\mathbb{R}$. As before, point (i) then follows from the dominated convergence theorem and Corollary 3.2. ∎

It is now easy to derive a simple proof of Theorem 1.1 in the case $a = 2$.

[2] At first sight, the application of Theorem 5.5.1 in [18] seemingly requires that $\gamma(n) :\equiv \frac{c-1}{2}$ be nonnegative (which is not true if $c = 0$). It appears this condition is stated merely for simplicity and, referring to the notation there, the proof of the theorem remains valid as long as $2\alpha\beta+\gamma\ge 0$ and $\gamma > -\alpha$.

Proof of Theorem 1.1, case $a = 2$. Recalling (13) and taking into account the change of variable (29), we have

$$\frac{1}{\sqrt{d_n}} I_q(B_E(S_p^n)) \sim \frac{\mathrm{e}^{-\frac{1}{p}-\frac{3}{4}}}{\sqrt{2\pi}} Z_{n,p}^{-1/d_n} \left(\mathbb{E}\,\langle \widetilde{L}_{n,p}, h_1\rangle^{\frac{q}{2}}\right)^{\frac{1}{q}},$$

where $h_1(y) := y$. Lemma 4.6.(ii) tells us that $Z_{n,p}^{-1/d_n} \sim 2\,\mathrm{e}^{\frac{3}{2p}}$, while Lemma 4.6.(i) gives

$$\lim_{n\to\infty} \mathbb{E}\,\langle \widetilde{L}_{n,p}, h_1\rangle^{\frac{q}{2}} = \langle \widetilde{\mu}_p, h_1\rangle^{\frac{q}{2}}.$$

Plugging in the value of $\langle \widetilde{\mu}_p, h_1\rangle$ given by Lemma 5.1, we find

$$\frac{1}{\sqrt{d_n}} \frac{\left(\mathbb{E}\,\|T\|_{\mathrm{HS}}^q\right)^{1/q}}{|B_E(S_p^n)|^{1/d_n}} \sim \mathrm{e}^{\frac{1}{2p}-\frac{3}{4}} \sqrt{\frac{p}{\pi(p+2)}}.$$

∎

4.3. **Asymptotics of c_n.** It remains to establish the asymptotics (7) for the coefficient c_n involved in Weyl's integration formulas. This is well known from the formulas given in [2], see for example [22, Lemma 3.3] in the self-adjoint case. We detail the proof for completeness.

Lemma 4.7 (Asymptotics of c_n)**.** *We have*

$$\sqrt{n} \cdot c_n^{1/d_n} \sim \mathrm{e}^{\frac{3}{4}} \sqrt{\frac{4\pi}{ab}} \quad \text{as } n \to \infty.$$

Proof. The coefficient c_n is related to the volume of the unitary group $U_n(\mathbb{F})$. Specifically, supposing first $a = 1$, we know after [2, Propositions 4.1.1] that

$$c_n = \frac{|U_n(\mathbb{F})|}{|U_1(\mathbb{F})|^n\, n!},$$

where, according to [2, Proposition 4.1.14],

$$|U_n(\mathbb{F})| = (2\pi)^{\frac{bn(n+1)}{4}} \cdot 2^{n(1-\frac{b}{2})} \Big/ \prod_{k=1}^{n} \Gamma\left(\frac{b}{2}\,k\right).$$

Since $\log\Gamma(z) = z\log z - z + o(z)$ as $z \to \infty$, we have

$$\log\Gamma\left(\frac{b}{2}k\right) = \frac{b}{2}k\log k + \frac{b}{2}\left(\log\frac{b}{2} - 1\right)k + o(k), \quad k \to \infty.$$

Therefore, using that $\sum_{k=1}^n k\log k = \frac{1}{2}n^2\log n - \frac{1}{4}n^2 + o(n^2)$ and $d_n \sim bn^2/2$, we find

$$\frac{1}{d_n}\log\prod_{k=1}^{n}\Gamma\left(\frac{b}{2}\,k\right) = \frac{1}{2}\log n + \frac{1}{2}\log\frac{b}{2} - \frac{3}{4} + o(1).$$

Hence

$$\frac{1}{d_n}\log c_n = \frac{1}{d_n}\log|U_n(\mathbb{F})| + o(1) = -\frac{1}{2}\log n + \frac{1}{2}\log\frac{4\pi}{b} + \frac{3}{4} + o(1),$$

which leads to the statement in the case $a = 1$. When $a = 2$, we have instead $d_n \sim bn^2$ and, after [2, Proposition 4.1.3],

$$c_n = \frac{|U_n(\mathbb{F})|^2}{|U_1(\mathbb{F})|^n\, n!}\, 2^{-\frac{bn(n-1)}{2}}.$$

So the difference with above is that, here,

$$\begin{aligned}\frac{1}{d_n}\log c_n &= \frac{2}{d_n}\log|U_n(\mathbb{F})| - \frac{1}{2}\log 2 + o(1)\\ &= -\frac{1}{2}\log n + \frac{1}{2}\log\frac{2\pi}{b} + \frac{3}{4} + o(1).\end{aligned}$$

The statement in the case $a = 2$ thus follows. ∎

Remark 4.8. *We can recover the asymptotic volumes of the Schatten unit balls. These were recently derived in [21, Theorem 3.1] and [20, Theorem 1], completing the much earlier computations of Saint-Raymond [37]. Indeed, starting from the expression of the volume of $B_E(S_p^n)$ in (9) and plugging in the asymptotics of $Z_{n,p}$ (Lemmas 4.2 and 4.6) and of c_n (Lemma 4.7), we find*

$$\left|B_E(S_p^n)\right|^{\frac{1}{d_n}} = (abn\gamma_p)^{\frac{1}{p}} \cdot \frac{c_n^{1/d_n}}{\Gamma\left(1+\frac{d_n}{p}\right)^{1/d_n}} \cdot Z_{n,p}^{1/d_n} \sim (abn\gamma_p)^{\frac{1}{p}} \cdot \frac{\mathrm{e}^{3/4}\sqrt{\frac{4\pi}{nab}}}{\left(\frac{d_n}{\mathrm{e}p}\right)^{1/p}} \cdot \frac{1}{2}\,\mathrm{e}^{-\frac{3}{2p}}.$$

Hence, because $d_n \sim abn^2/2$,

$$\lim_{n\to\infty} n^{\frac{1}{2}+\frac{1}{p}}\,|B_E(S_p^n)|^{\frac{1}{d_n}} = (2p\gamma_p)^{\frac{1}{p}}\,\mathrm{e}^{\frac{3}{4}-\frac{1}{2p}}\sqrt{\frac{\pi}{ab}}.$$ ∎

5. Properties of the equilibrium measures

In this section we gather some auxiliary results on the equilibrium measure μ_p of Lemma 4.1. We first carry out some easy computations. Next, we establish the regularity of the solution to the master equation occurring in the proof of Proposition 4.3.

5.1. Some integral computations.

We recall the notation $h_2(x) := x^2$, $h_1(y) := y$, and the equilibrium measures μ_p and $\widetilde{\mu}_p$, see Lemmas 4.1 and 4.5. We further denote by $\varrho(\mathrm{d}x) := (\pi\sqrt{1-x^2})^{-1}\mathbb{1}_{\{|x|<1\}}\mathrm{d}x$ the standard Arcsine distribution on $[-1,1]$.

Lemma 5.1. *We have*

$$\langle \mu_p, h_2\rangle = \langle \widetilde{\mu}_p, h_1\rangle = \frac{p}{2p+4}.$$

Proof. First, $\langle \mu_p, h_2\rangle = \langle \widetilde{\mu}_p, h_1\rangle$ because $\widetilde{\mu}_p$ is the image measure of μ_p by h_2. Next we know [3, Lemma 4.1] that μ_p is the distribution of AB where A and B are independent variables with A following ϱ and B following the $\mathrm{Beta}(p,1)$ distribution. We easily conclude that

$$\langle \mu_p, h_2\rangle = \mathbb{E}\,A^2 \cdot \mathbb{E}\,B^2 = \frac{1}{2}\cdot\frac{p}{p+2}.$$ ∎

Lemma 5.2. *We have*

$$\frac{1}{\pi^2}\int_{-1}^{1}\int_{-1}^{1}\frac{(x+y)^2(1-xy)}{\sqrt{1-x^2}\sqrt{1-y^2}}\,\mathrm{d}x\mathrm{d}y=\frac{1}{2}.$$

Proof. Let X, Y be independent variables with law ϱ. Then the left-hand side is

$$\begin{aligned}\mathbb{E}\,(X+Y)^2(1-XY)&=\mathbb{E}\,(X+Y)^2-2\,\mathbb{E}\,X^2\,\mathbb{E}\,Y^2-\mathbb{E}\,X^3\,\mathbb{E}\,Y-\mathbb{E}\,X\,\mathbb{E}\,Y^3\\&=\frac{1}{2},\end{aligned}$$

using that $\mathbb{E}\,X=\mathbb{E}\,Y=0$ and $\mathbb{E}\,X^2=\mathbb{E}\,Y^2=\frac{1}{2}$. ∎

5.2. **Regularity of the solution to the master equation.** We establish the regularity of the solution ψ_p to the master equation

$$\Xi_p\psi=\frac{1}{2}h_2+c_{h_2}, \tag{30}$$

where

$$\Xi_p\psi(x):=-\frac{1}{2}V_p'(x)\psi(x)+\int\frac{\psi(x)-\psi(y)}{x-y}\,\mu_p(\mathrm{d}y),\qquad x\in\mathbb{R}.$$

A general expression in terms of the test function and of the equilibrium measure is given in [8, Lemma 3.3], see also Section B.5 there. The inversion of the master operator was first achieved in [7, Lemma 3.2]. In our framework, the expression of ψ_p can be made quite explicit: first, we compute $c_{h_2}=-1/4$, and

$$\psi_p(x)=\begin{cases}-\dfrac{x\sqrt{1-x^2}}{2\pi f_p(x)}, & \text{if } |x|\le 1,\\[2ex] -\dfrac{|x|\sqrt{x^2-1}}{2\zeta_p'(x)}, & \text{if } |x|>1,\end{cases} \tag{31}$$

where

$$f_p(x):=\frac{px^{p-1}}{\pi}\int_x^1\frac{v^{-p}}{\sqrt{1-v^2}}\,\mathrm{d}v, \tag{32}$$

is the Lebesgue density of μ_p which we have already seen in Lemma 4.1.(ii), and ζ_p' is the odd function given for $x>1$ by

$$\begin{aligned}\zeta_p'(x)&:=\frac{\mathrm{d}}{\mathrm{d}x}\left(-\int\log|x-y|\,\mu_p(\mathrm{d}y)+\frac{1}{2}\,V_p(x)\right)\\&=px^{p-1}\int_1^x\frac{v^{-p}}{\sqrt{v^2-1}}\,\mathrm{d}v,\end{aligned} \tag{33}$$

see [36, pp. 240-241]. We further observe the similarity between (32) and (33): applying the change of variable $t\leftarrow(1-v)/x$, we have, for every $0<x<1$,

$$\begin{aligned}\pi f_p(1-x) &= \frac{p(1-x)^{p-1}}{\pi}\int_{1-x}^1 \frac{u^{-p}}{\sqrt{1-u^2}}\,\mathrm{d}u \\ &= p(1-x)^{p-1}\sqrt{x}\int_0^1 \frac{(1-xt)^{-p}}{\sqrt{2-xt}}\,\frac{\mathrm{d}t}{\sqrt{t}} \\ &= \sqrt{x}\,w_p(-x),\end{aligned}$$

where we have set

$$w_p(u) \coloneqq p(1+u)^{p-1}\int_0^1 \frac{(1+tu)^{-p}}{\sqrt{2+tu}}\,\frac{\mathrm{d}t}{\sqrt{t}}, \qquad u \geq -1. \tag{34}$$

Similarly, we can see that $\zeta_p'(1+x) = \sqrt{x}\,w_p(x)$ holds for all $x > 0$. Because ψ_p is odd, we conclude that the solution $\psi_p \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$ to the equation (30) is more simply expressed by

$$\psi_p(x) = -\frac{x\sqrt{1+|x|}}{2w_p(|x|-1)}, \qquad x \in \mathbb{R}. \tag{35}$$

We denote by $\lceil p \rceil$ the smallest integer greater than or equal to p.

Lemma 5.3 (Regularity of ψ_p)**.** *For every $p \in (1,\infty)$, the function ψ_p is of class $\mathcal{C}^{\lceil p\rceil - 1}$. In particular, it is of class $\mathcal{C}^3$ when $p > 3$.*

Proof. Since ψ_p is odd, it suffices to check the regularity on $[0,\infty)$. It is plain that the function w_p in (34) does not vanish on $(-1,\infty)$ and, by differentiation under the integral sign, that it is of class $\mathcal{C}^\infty$ there. Given (35), it is then clear that ψ_p is of class $\mathcal{C}^\infty$ on $(0,\infty)$. It remains to check that ψ_p has $\mathcal{C}^{\lceil p\rceil-1}$-regularity at 0^+.

Because of the expression (31) and of $x \mapsto \sqrt{1-x^2}$ being smooth at 0, the regularity of ψ_p at 0 is equivalent to that of $\bar{\psi}_p(x) \coloneqq x/f_p(x)$. For $0 < x < y < 1$, the Taylor series

$$(1-v^2)^{-\frac{1}{2}} = \sum_{k=0}^\infty \frac{\binom{2n}{n}}{4^n}\,v^{2n}$$

is uniformly convergent on $[x,y]$, and expanding in (32) gives

$$f_p(x) = \frac{px^{p-1}}{\pi}\limup_{y\uparrow 1}\int_x^y \frac{v^{-p}}{\sqrt{1-v^2}}\,\mathrm{d}v = \frac{p}{\pi}\sum_{n=0}^\infty \frac{\binom{2n}{n}}{4^n}\,x^{p-1}\int_x^1 v^{2n-p}\,\mathrm{d}v,$$

by the monotone convergence theorem. The latter integral equals $-\log x$ if $p = 2n+1$, and $(x^{2n+1-p}-1)/(p-2n-1)$ otherwise. In any case, there exist constants $A, B \in \mathbb{R}$ (possibly zero) such that

$$g(x) \coloneqq f_p(x) + Ax^{p-1}\log x + Bx^{p-1} = \frac{p}{\pi}\sum_{\substack{n\geq 0\\ 2n+1\neq p}} \frac{\binom{2n}{n}}{4^n(p-2n-1)}\,x^{2n}$$

is decomposed as a power series with radius 1, so g defines a function of class $\mathcal{C}^\infty$ at 0. Because, for $p > 1$, $Ax^{p-1}\log x + Bx^{p-1}$ has $\mathcal{C}^{\lceil p\rceil-2}$-regularity at 0^+, f_p is therefore of class $\mathcal{C}^{\lceil p\rceil-2}$ at 0^+. Then so is $1/f_p$ since $f_p(0) = g(0) = p\pi/(p-1) \neq 0$, and we conclude by Leibniz formula that $\bar{\psi}_p$ (and thus ψ_p) is of class $\mathcal{C}^{\lceil p\rceil-1}$ at 0^+. ∎

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